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ABSTRACT

From Aristotle up to the Baroque knowledge was considered as essentially determined by its object. Since Kant and his so-called *Copernican Revolution of Epistemology* the epistemic subject and the method and means of its activities become equally, or even more important. Mathematics is, first of all, an activity, which has increasingly liberated itself from metaphysical and ontological agendas. As a consequence, the sense of mathematical symbolizations acquired partial independence from reference. The theories of modern science are schemes of interpretation of objective and socio-historical reality, rather than images of it. To observe the dynamics of the complementarity of sense and reference of symbolic representations becomes an important way of all knowledge related research. Perhaps more than any other practice, mathematical practice requires a complementarist approach, if its dynamics and meaning are to be properly understood.

Keywords: The complementarity of existence and meaning. Instrumentalism. The difference between concepts and functions.

RESUMO

De Aristóteles até o Barroco, o conhecimento era considerado essencialmente determinado por seu objeto. Desde Kant e sua chamada Revolução Copernicana da Epistemologia, o sujeito epistêmico e o método e os meios de suas atividades tornam-se igualmente, ou até mais importantes. A matemática é, antes de tudo, uma atividade que se libertou cada vez mais das agendas metafísicas e ontológicas. Como conseqüência, o senso de simbolizações matemáticas adquiriu independência parcial de referência. As teorias da ciência moderna são esquemas de interpretação da realidade objetiva e sócio-histórica, em vez de imagens dela. Observar a dinâmica da complementaridade de sentido e referência de representações simbólicas torna-se um caminho importante de toda pesquisa relacionada ao conhecimento. Talvez mais do que qualquer outra prática, a prática matemática requer uma abordagem complementarista, se a sua dinâmica e significado devem ser adequadamente compreendidos.

Palavras-chave: A complementaridade de existência e significado. Instrumentalismo. A diferença entre conceitos e funções.

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INTRODUCTION

The distinguished mathematical author Reuben Hersh is certainly not alone, when he considers the question, “In what sense do mathematical objects exist?” (Hersh, 1997) the main problem in the philosophy of mathematics. Until about 1800 western philosophy believed, he says, that there were two kinds of things in the world: mental and physical. Hersh thinks that mathematics shows the inadequacy of this belief and he proposes instead to consider mathematical objects as social entities and to acknowledge that mathematics is essential a social reality. Hersh wants to avoid the alternative of mentalism vs. empiricism. Social entities, he says, “are neither mental nor physical”, but they have “mental and physical aspects” (Hersh, 1997, p. 14). This third road between mentalism and empiricism is characterized by an attention on semantic categories, by what we say and how we say it.

Mathematics is a symbolic activity. A symbol has relations to other symbols or to some existents which it symbolizes. The relations of a symbol to other symbols are called its *intension* (or sense) while its references to objects of some kind are called its *extension*. The complementarity between intension and extension is crucial in understanding the evolution of knowledge and it comes under various names: intension/extension, sense/reference, meaning/information, identity/difference, analytic/synthetic.

Language and mathematics form different contexts of human thought and action. Questions of meaning dominate in language, while problems of reference or existence dominate in mathematics and in the philosophy of mathematics (Hersh). Kurt Gödel said: “The subject matter of logic is intensions (concepts); that of mathematics is extensions (sets)” (Gödel, in: Wang, 1996, p. 247). Therefore, we should have a concept theory and a theory of mathematical reference as well.

One should observe that the concept of sense or meaning designates the principle of order of human experience and feeling. For all human sciences, “meaning” is the central concept, because language has no other task than the communication of “meaning”. This insight led already in ancient times to hermeneutics, that is, the art of interpretation, which was applied first to religious texts. Mathematics and science, on the other hand, deal with objectivity and truth.

In the recognition of knowledge as a symbolic activity three elements appear therefore, the ideas, the syntax or grammar of symbolic operation and third the application or interpretation. Charles S. Peirce (1839-1914), son of a well-known mathematician and America's greatest philosopher, defines a sign as a tripartite entity:

"A representation is that character of a thing by virtue of which, for the production of a certain mental effect, it may stand in place of another thing. The thing having this character I term a representamen, the mental effect, or thought, its interpretant, the thing for which it stands, its object".

A sign or representamen, according to Peirce, "stands for that object, not in all respects, but in reference to a sort of *idea*, …. Idea is here to be understood in a sort of Platonic sense, very familiar in everyday talk" (Peirce, CP 1.564). The object of a sign is not necessarily something empirically given. Otherwise it would be difficult to apply such semiotic concepts to mathematics. The object, as in the examples of mathematical entities or in case of theoretical terms, like energy – of which heat and motion are different representations, for instance, – or the general triangle, is not necessarily given as such, but is rather a universal object or a hypostatic abstraction.

While the classical formula portrays the sign in terms of a dyadic relationship, the Peircean definition conceives of it in terms of a triadic structure. The fundamental triad in Peircean semiotics is precisely "object — sign (idea) — interpretant" (Peirce, CP 8.361).

I.

Everybody agrees that mathematics distinguishes itself from the rest of the sciences by the conception of reliable proof. And there is in general unanimity or consensus of the informed with respect to whether some mathematical truth has in fact been established by valid proof or not. Proof is, however, also intended to be a vehicle of introducing the student or newcomer to mathematics and the mathematical way thinking, and therefore it has been made more and more explicit and rigorous since the 19th century and has been formalized until finally questions of meaning and development have been lost sight of. Formal proof seems absolutely reliable, but it is not quite certain any more "what it is reliable about", as Lakatos once said, paraphrasing a well-known *bonmot* by Bertrand Russell.

But, how should we be able to learn something new by such kinds of proofs? It seems that there is a fundamental difference between following the course of an argument, on the one side, and understanding it, on the other. Suppose I have constructed a proof for some mathematical theorem, which after having checked out the argument step by step, seems now completely clear to me.

> Suppose that a great authority announces that there is something wrong with the argument. In that case my experience upon checking over the argument may be quite different from what it was before this announcement was made. Just as before, I find that the argument appears to be correct; only this time I do not accept it as being correct.
> (Stolzenberg, 1980, p. 263)

Formal operations mean exactly themselves and refer to nothing else. This situation prevails even if I cannot find fault with my argument. The distinction between being correct and merely appearing to be correct is exactly the same as that between seeing something and merely following a rule or a chain of arguments. What is missing is exactly that third element, namely a reference to something objective.

According to the Aristotelian conception of proof the knower's and the listener's status were at least as important as that of the object to be known. It was not sufficient to give something as a dead thing or strictly individual existent. Aristotle would never separate science or mathematics from philosophy, and it has been claimed indeed that to understand his work one would have to consider "the four discourses": Poetics, Rhetoric, Dialectics and Logic as but variants of only one unified science.

We are not suggesting, to restore the Aristotelian world view. That would be a futile undertaking – but it is worthwhile to remember that the distinguished mathematician, winner of the Fields medal for the year 1958, R. Thom did criticize the New Math movement from such an Aristotelian perspective. Thom writes: "In 1969 I wrote an article against the use of set theory in school… For instance, they put in front of a child a box containing some cubes… And they asked the child to take out from the box the cubes that were large or blue. In usual language the copula "or" cannot be between two adjectives, which do not belong to the same genus… The opposition of large and small occurs in the genus of quantity, whereas the opposition between blue and red is an opposition in the genus color" (p. 9).

This criticism of formalism led Thom to consider the old logic of Aristotle and to look at his metaphysics, concluding:

> I think that Aristotle had the idea that logic has to be founded in the ontological nature of the concepts. It has to express something in reality. …. The concept has a sort of ontological ground in reality, which is its substantial form, its essence. … So if we want to use logic in a natural way which is founded in reality, we have to introduce this sort of essential quality which is associated with any concept, and we have accidental references of this concept which modify its quality. If we take this point of view, the syllogism fails most times.
>
> (Thom, 1992, p. 8)

Nature and the universe of things do determine neither our actions nor our thoughts or interpretations. But from Aristotle to Leibniz concepts had in fact been considered collections of attributes that expressed the essence of things. The concepts, which are Aristotle's special object and interest, are the generic concepts of the descriptive and classifying natural sciences. For Aristotle things had essences, later words have meanings. Knowledge was dominated by the intuition of its objects until to the Scientific Revolution of the 16th/17th centuries. The essence of the olive-tree, the horse, the lion, is to be ascertained and established.

Wherever Aristotle leaves

> The field of biological thought, his theory of the concept at once ceases to develop naturally and freely. From the beginning, the concepts of geometry, especially, resist reduction to the customary schema. …. Mathematical concepts, which arise through genetic definition, through the intellectual establishment of a constructive connection, are different from empirical concepts, which aim merely to be copies of certain factual characteristics of the given reality of things. …. In the definitions of pure mathematics … the world of sensible things is not so much reproduced as transformed and supplanted by an order of another sort. If we trace the method of this transformation …. an ordered system of strictly differentiated intellectual functions are revealed, such as cannot even be characterized, much less justified by the simple schema of *abstraction*.
>
> (Cassirer, 1953, p. 12-14)

Indeed, Euclidean geometry was governed by a very different limitation, namely the requirement of constructability with compass and ruler. Geometry became the first science in the struct sense of today, because the context of its activities and knowledge claims was limited and over-seeable. Everything that could constructed with compass and ruler in a finite number of steps starting from postulates, which licensed certain basic constructions, like: “To draw a straight line from any point to any point”, or: “A circle can be drawn with any point as its centre and with an arbitrary radius”, becomes a legitimate object of geometrical knowledge.

And arithmetic and algebra had their own limitations by requiring the feasibility of algebraic operations. Even negative numbers have long been illegitimate items. Algebra requires a relational thinking. So the question: relations between what? Students who are able to solve the equation “3+5 = ?” without difficulties, in general cannot solve the other equation “3+? = 8”.

Then came the time when the request of logical consistency became considered the only legitimate limitation of mathematical productivity. Pure mathematics is established by the idea “that intelligible means identical or as the modern logicians say, the highest principle of thought is the principle of tautology” (Mouy, 1971, p. 50).

Now the fundamental question was: How to prove the consistency of mathematics?

Gödel’s incompleteness theorem says that it is impossible.

Therefore, Gödel’s incompleteness theorem furnishes the easiest way to recognize the *indispensability* of a complementarity approach to mathematics.

In fact, Kurt Gödel demonstrated that within any given branch of mathematics, there would always be some propositions that couldn’t be proven either true or false using the admitted rules and axioms. One might be able to prove every conceivable statement by going *outside* the system in order to come up with new rules and axioms, but by doing so you will only create a larger system with its own unprovable statements. Therefore *all* logical system of any complexity are, by definition, incomplete; each of them contains, at any given time, more true statements than it can possibly prove according to its own defining set of rules. Mathematics cannot be established as a purely formal system but requires some intended applications or models.

Kant was the first to see this clearly and Gödel recognized Kant’s merits. The separation of mind and matter stimulated the search for certainty of knowledge and for secure methods to achieve such knowledge. Descartes and Leibniz had missed to realize this and tried to base knowledge on purely conceptual means.

In a letter to Arnauld of July 14, 1686, Leibniz writes, for example: “Regarding the subject of metaphysics I claim to advance by geometrical demonstrations, positing only two primary truths; to wit, in the first place, the principle of contradiction, (for if two contradictories could be true at the same time all reasoning would be useless); and secondly, the principle that nothing is without reason, or that every truth has its proof a priori, drawn from the meaning of the terms, although we have not always the power to attain this analysis".

The merit of Kant's doctrine lay exactly in his assertion that existence is a necessary prerequisite of consistency claims. The principle of consistency, according to Kant, only applies if there is an object given. The statement that "a triangle has three angles", says Kant, "does not enounce that three angles necessary exist, but upon the condition that a triangle exists three angles must necessarily exist in it" (Kant, B 622).

But Kant still believed that whatever belongs to conditions of knowledge and experience cannot be its object. And this was his fundamental error that separated him from the evolutionary perspective which dominates modernity. Modern mathematics is becoming meta-mathematics in a very productive way, as Gödel has shown.

Gödel's theorem may be made plausible in various ways. One way consists in using arguments from *algorithmic information theory*. In an information-theoretic approach to Gödel's theorem one argues "that if a theorem contains more information than a given set of axioms then it is impossible for the theorem to be derived from the axioms" (Chaitin, 1987, p. 55).

A similar situation is the following: one can demonstrate that "nearly all" real numbers are not computable, given Turing`s hypothesis, because the set of computable numbers is enumerable. But this set itself is not even computable, otherwise Cantor`s diagonal method would produce a contradiction. And given a real number, an infinite decimal fraction, one cannot in general decide whether it is random or computable.

In mathematics formal axiomatic methods, which involve no existence claims whatsoever are indispensable, how could the real numbers otherwise be characterized, for example. But axiomatic theories cannot guarantee to provide real knowledge, no more than the algorithmic search can guarantee an exit from a given maze, because it cannot guarantee the very existence of an exit. As a consequence, one has, in addition, to accept the existence of non-conceptual knowledge and experience of objects. Mathematics is, however, no empirical science like physics or biology. Therefore, we have to create model worlds. The proof of the impossibility of doubling the cube with Euclidean means, became possible as soon as people modelled the geometrical constructions in arithmetical terms, creating the notion of "constructible number" and finally showed than the third root of 2 was not a constructible number.

Russell's paradoxes provide another argument in favor of the complementarity of concepts (meanings) and objects, as we had defined it, because the paradoxes of set theory arose "not from an inconsistency in our intuitive notion of set, but from a conflation of two notions: *set-as-one* vs. *set -as-many*" (Potter, 1990, p. 10). And this means nothing but conflating *concepts* and *objects*. A concept is what a set of objects have in common, like the set of red cows. But the set of red cows is not a red cow. Concepts are not objects, or at any rate, they are not objects of the same type as the objects falling under them. Russell asked for the collection of all sets that are not a member of themselves and this would have to be, if it existed, the set of all sets, something impossible.

II.

Let us try and dedicate some more reflections to the concerns of Rene Thom, trying to indicate the essential changes in our notions of concept and proposition since the days of Plato and Aristotle. We shall concentrate on the *Scientific Revolution* of the 16th/17th century and in particular on Leibniz (1646-1716), because the Leibnizian project, being somewhat ambiguous and with a Janus head "is situated at the very heart of the classical thought" (Foucault, 1973, p. 57).

Michel Foucault has described this revolution as a transition from the Renaissance, an age of interpretation to the Baroque, an epoch of representation and symbolization. In his "*The Order of Things*" Foucault writes:

> At the beginning of the 17th century writing has ceased to be the prose of the world, resemblances and signs have dissolved their former alliance, similitudes have become deceptive… Thought ceases to move in the element of resemblance. Similitude is no longer the form of knowledge, but rather the occasion of error... 'It is a frequent habit', says Descartes, in the first lines of his *Regulae*, 'when we discover several resemblances between things, to attribute to both equally, even on points in which they are really different, that which we have recognized to be true of only one of them'. The age of resemblance is drawing to a close... And just as interpretation in the sixteenth century… was essentially a knowledge based upon similitude, so the ordering of things by means of signs constitutes all empirical forms of knowledge as knowledge based upon identity and difference.
> (Foucault, 1973, p. 47-51 and p. 56-57)

Descartes has been the first person to realize that the essence of mathematics lies in the combination of calculation, on the one hand, and of perception and geometric objectivity, on the other hand. Descartes had in 1619 already tried to design a program and a method by which the problems of continuous and discrete magnitude could be treated analogically. He programmatically outlines the new ideas in an important letter to Beeckman of March 26, 1619 (Shea, 1991, p. 44). Descartes was a revolutionary and the preconditions to the possibility of Cartesian thinking were two: "First the collapse of the rigid distinction between (productive) arts and (theoretical) sciences and, second, the flourishing of the practice of symbolization" (Lachterman, 1989, p. 125).

Leibniz took up the second aspect of Descartes' orientation, but not the first. Leibniz's attitude was quite ambiguous. He was a modern, a scholastic and a Renaissance philosopher, not just at different stages but even at the same time. In one sentence he was an early modern in whose assumptions and procedures residually scholastic and Renaissance elements can still be identified" (Brown, 1993, p. 215).

In contrast to the radicalism of Descartes, Leibniz was

> ingeniously conservative. The merit of the old system (of Aristotelian demonstrations, our insertion) was that it gave us some understanding of the nature of the interconnection of truths... Leibniz grafted a new methodology on to the old theory of demonstration… In the old tradition only universal propositions are subject to demonstration. In the new practice only what we now call pure mathematics fits this model. But Leibniz making proof a matter of ontology, not methodology asserts that all true propositions have an a priori proof, although in general humans cannot make this proof".
>
> (Hacking, 1986, p. 221)

Leibniz has a very distinctive notion of truth, one which underlies much of his metaphysics. But this notion of truth goes back to Aristotle's *Organon* (cf. *Posterior Analytics* I.4), as Leibniz himself says. As Leibniz puts it in a letter to Arnauld, "in every true affirmative proposition, whether necessary or contingent, universal or particular, the notion of the predicate is in some way included in that of the subject. *Praedicatum inest subjecto;* otherwise I do not know what truth is" (Kneale & Kneale, 1971, p. 323). Every truth has an a priori proof.

In today's mathematics and mathematical logic, only the truth value of propositions counts in inference, that is, the fact that one cannot infer anything wrong from truth, such that the statement *"if A then B"* comes out false only if *A* is true and *B* is false.

Example: “(A: x bigger than 5) implies (B: x is bigger than 3)”. Modern mathematic and logic are truth preserving machines, but do not produce any new truths. Leibniz wanted more and hence his view that the world is a collection of substances, that is, concepts, rather than things.

When considering the content of a statement as aggregate of concepts, proofs make sense only as soon as a preestablished harmony between the realms of concepts and objects is assumed. However, even mathematical truths depend on the distinction between concept and object. For example, two geometrical figures are identical or the same as existing things only if they are made up of the same points in space, while geometrical equality denotes what the mathematician understands by the concept of “congruence”. Congruence constitutes the geometric figures as concepts, not as objects.

However, Leibniz defines equality by his “*Principle of the Identity Indiscernibles*”, according to which two things are equal if they agree in all attributes, that is, if their “complete concepts” coincide. Leibniz does not accept space as a set of points. This provided an insurmountable obstacle to Leibniz’ endeavors to make proofs purely formal entities. For example, the proposition “equality of content implies congruence” is true in the one-dimensional case, but wrong in plane geometry. Two triangles could measure the same area and still be incongruent. This is different for Leibniz, because congruence is the basic geometric equality (which, however, does not allow one to build a calculus) (Otte 2014, p. 271).

Leibniz problem and his failure becomes clearer as soon as we compare him with Bolzano. Let us see how Bernard Bolzano (1781-1848) the most important logician between Leibniz and the 19th century, deals with the problems involved. In his 1837 *Wissenschaftslehre* Bolzano attempted to provide logical foundations for all sciences, building on an analysis of the semantic of human communication.

Bolzano agreed with Kant`s rejection of a *pre-established harmony* between mind and matter, between our cognitions or representations and the objective world in the sense of Leibniz. “It had exactly been Bolzano, who … had completely anti-platonically distinguished between the structure of being and the structure of cognition” (Neemann, 1972, p. 81).

The meaning of our representations is some <u>third</u> between object and definition or sign. Contemporaneous authors had argued that if one defines a triangle by saying it is a geometrical figure "which has an angle-sum of 180 degrees", then the theorem about the angle-sum of a triangle would come out analytic.

> I think differently here, responds Bolzano, as I do not consider a proposition a mere conjunction of words, but intend it as the sense of the statement, I do not admit that the proposition remains the same if one assigns to the word *triangle* at times this one and at other times a different concept. Such habits would be analogous to our pronouncing the proposition: 'Euclid was a famous mathematician' intending by the name Euclid at one moment of time Ptolemäus teacher of geometry in Alexandria and at a different moment think of Euclid of Megara the student of Socrates… In order to distinguish propositions from one another it suffices that they consist of different representations (Vorstellungen) even though they might refer to the very same objects.
> (Bolzano, WL §148)

And on this distinction the other one between analytic and synthetic propositions is crafted, because it has made both Kant as well as Bolzano, aware of the errors of the traditional notion of a concept as something established by abstraction, wherefrom results the law of inverse relation between content and extension of concepts. Bolzano, refusing this law of inverse relationship, writes:

> If I am so fortunate as to have avoided a mistake here which remained unnoticed by others, I will openly acknowledge what I have to thank for it, namely it is only the distinction Kant made between analytic and synthetic judgments, which could not be if all of the properties of an object had to be components of its representation.
> (Bolzano, WL, §120)

A proposition is obviously synthetic if its predicate contains a characteristic of the object, that is not already part of the presentation of the subject of that proposition. All our thinking occurs by means of signs and there is no intuitive knowledge in the strict sense. From such observations results Bolzano's definition of analytical propositions, which is as follows:

> If there is a single representation (eine einzige Vorstellung) in a proposition which can be arbitrarily varied without disturbing its truth or falsity… then this character of the proposition is sufficiently remarkable to distinguish it from all others. I permit myself thence to call propositions of this kind, borrowing an expression from Kant, *analytic*, all others, however, *synthetic* propositions.
> (Bolzano, WL §148)

The sentence, “Socrates is mortal” is synthetic, if God exists, for example, and is analytic otherwise. It might thus occur that we do not know, whether a proposition is analytic or not (Kneale & Kneale, 1971, p. 366). It depends on the kind of world we live in. The realms of concepts and objects are not coordinated, as Leibniz had believed, making both characteristics of God’s mind.

That certain propositions, like “this flower smells pleasant”, or “a bottle of wine costs 10 thaler” appear as sometimes true and sometimes false, depending on circumstances, is due to our disregarding that the proposition in question does not remain the same. “This”, for example, is an indexical sign with different referents depending on context. And in the second example we assume, says Bolzano, tacitly that there is a context of time and space when we hear somebody making such a judgment (Bolzano, WL, §147). Hearing somebody say such a sentence we automatically would ask: where? And: when?

Outside actual discourse pragmatics the proposition above – “a bottle of wine costs 10 thaler” should read: “At place X and time Y, a bottle of wine costs 10 thaler”. If it would remain true forever and everywhere in the world that “a bottle of wine costs 10 thaler” then this proposition would be analytic, according to Bolzano`s definition of analyticity. Leibniz views are similar, but with the essential difference that Leibniz claims to be able to state a priori which positions are analytic and which not. The world is just an idea in God’s mind such there are no facts independent from Gods ideas and logical reasonings. However even geometry or arithmetic might surprise us, something Leibniz did not take into a account.

As a consequence of this separation between concepts and things, ideas and facts, the notion of scientific concept and of proposition had to be reconstructed in new ways. With respect to the form of a proposition it was a decisive insight of Bernard Bolzano, Charles S. Peirce (1839-1914) and Gottlob Frege (1848-1925) that a judgment or proposition is not an aggregate of terms that represent concepts or universals, but that its elements have different kinds of roles in this context. One of those basic roles is that of denoting existents. This is important because the scientist or mathematician might try and explore the nature of things that he does not yet know. Think of the famous unknown “x” of formal algebra.

Peirce characterizes the subject of a proposition generally as either general or particular. A general or “universal subject is one which indicates that the proposition applies to whatever individual there is in the universe or to whatever there may be of a general description without saying that there is any. A particular subject is one which does not indicate what individual is intended further than to give a general description of it, but does profess to indicate an existent individual at least" (Peirce, CP 2.324).

And concerning concepts, one observes that they were no more conceived as the result of abstraction, but had become "constructed" functions of cognitive activity (Cassirer 1953, p6ff). And functions are assembled like propositions out of two incongruent elements: arguments and relations: y = f(x)!

III.

In his invited lecture to the Second International Congress on Mathematics Education in Exeter in 1972 Thom had said:

"The real problem which confronts mathematics teaching is not that of rigor, but the problem of the development of meaning, of the existence of mathematical objects" (Thom, 1973, p. 204).

Thom considers neither abstract set-theory (see I.) nor the structuralism a la Bourbaki or Piaget suitable for introducing modern mathematics into the school. As was said, since Aristotle, all knowledge was considered as being determined by its object, and mathematics was generally defined as a theory of quantities. However, language and the semantics of communication or semiotics are beginning to play an increasingly important role, since the 17th century. Already in 1710, Berkeley had objected against Locke's idea of a *general triangle*. Such ideas, Locke had cautioned

> are not so obvious as particular ones… For example, does it not require some pains and skill to form the general idea of a triangle for it must be neither oblique, nor rectangle, equilateral, equicrural, nor scalenon; but all and none of these at once.
>
> (Locke, 1690, book IV, chap. 7, Section 9)

Now Berkeley asked the readers of Locke's *Essay concerning Human Understanding* to try and find out whether they could possibly have "an idea that shall correspond with the description here given" (Berkeley, 1989, p. 70). And to this logical dilemma Berkeley proposed a "representational" solution, saying that

> we shall acknowledge that an idea, which considered in itself is particular, becomes general, by being made to represent or stand for all other particular ideas of the same sort. …. Universality, so far as I can comprehend, is not consisting in the absolute positive nature or conception of anything, but in the relation it bears to the particulars signified or represented by it.
>
> (Berkeley, 1989, p. 70)

A particular geometrical idea, like line or triangle "becomes general by being made a sign" (ibid.). What we perceive, in geometry as elsewhere, are signs and these signs represent classes of things, rather than particular things. Jesseph has characterized Berkeley's philosophy of geometry by the term "representative generalization" and he writes: "The most fundamental aspect of Berkeley's alternative is the claim that we can make one idea go proxy for many others by treating it as a representative of a kind" (Jesseph, 1993, p. 33).

On such an account, a general triangle is a free variable, like the terms in axiomatic descriptions, and not a collection of determinate triangles. And which properties are essential to a "general triangle" depends on context, on the activity and its goals. If the task, for instance, is to prove the theorem that the medians of a triangle intersect in one point, the triangle on which the proof is to be based can be assumed to be equilateral, without loss of generality – because the theorem in case is a theorem of affine geometry and any triangle is equivalent to an equilateral triangle under affine transformations. This fact considerably facilitates conducting the proof.

What we see is that the object becomes now determined in accordance with the means of cognitive and communicative activity, rather than the other way around. In this sense Hersh is certainly right considering mathematical objects as social entities and to acknowledge that mathematics is essential a social reality.

People had understood from early on, that meanings could be universalized or synchronized only as soon as they are transformed into formal rules and functions. For example, one realizes that there is no unanimous consent about the nature of numbers even to-day. Platonists quarrel with empiricists or with formal and instrumental attitudes. But if one says, take 3 apples and add 5 more to them, everybody agrees on what that means.

The *New Math Reform* of the 60ties and 70ties of the 20th century wanted more and it gloated at the affirmation that for the first time in the history of mathematical education one was to succeed in reconciling "intuition and logic" (G. Papy 1967, oral presentation at the *Duesseldorf Academy of Sciences*) or fundamental theoretical ideas and concrete meanings by means of the set theoretical metaphor.

Modern mathematics had hoped to diminish the distance to mathematical science through set-theoretical ontology. Even if students were not to become professional mathematicians, mathematics at school could not be completely disconnected from mathematical science. One must ask whether school can afford to remain at a picture of mathematics of 18th-century Enlightenment.

Diderot had actually already in 1754 considered the mathematics of his time as finished and completed and without further interest. *On the interpretation of Nature* (French: *Pensees sur l'interpretation de la nature*) is a 1754 book, written by him in which he predicted that mathematics had reached its zenith and that a hundred years hence there would not be three great mathematicians in the whole of Europe. Mathematicians, Diderot writes, “like to criticize other thinkers of being metaphysicians, but increasingly chemists, physicists, and naturalists are directing the same criticism at them.

> A great upheaval in the sciences is imminent. In view of the present aspiration of the great minds, … I should almost like to claim that there will not be three great mathematicians in Europe within a century. This science will suddenly remain fixed to the spot where the Bernoullis, Euler, Maupertius, Clairaut, Fontaine, D’Alembert, and Lagrange have left it.
>
> (Diderot, 2000, p. 421)

Similar statements have come down to us from Lagrange.

The general feeling at the end of the 18th century was, that the traditional means of justification and organization of knowledge became as insufficient as the given methods showed themselves as incapable of producing new results. And this led in fact to the revolution of pure mathematics half a century later. As a result, knowledge and mathematical knowledge, in particular, were no longer defined in terms of its objects but became characterized in terms of its methods.

> No matter how much Lagrange may assert and insist that a function is for him an abstract object, in his thought patterns it somehow is residually a mechanical orbit or perhaps a physical function of state, whereas in Cauchy orbits and forces and pressures are always functions, as they are for us today.
>
> (Bochner, 1974)

And these new methods were of two different types: the approach of formal axiomatic stood in opposition o the movement of arithmetical rigor. Gottfried Martin writes:

> Purely historically mathematics can be divided into arithmetic and geometry. Aristotle recognized the significance of this division for the ontological problems of mathematics. A second kind of distinction is found by dividing mathematical propositions into principles and theorems. I will call the view, which supposes that there are unprovable principles *axiomatic*. I will call the opposing view *logicist.* This implies that the view that geometry is axiomatic is completely compatible with the view that arithmetic is logicist. The opposite point of view is hardly conceivable.
>
> (Martin, 1985, p. 6)

To this division in the concept of mathematics corresponded a contrast in the conceptions of logic, as Martin indicates *Logic as a universal Language* (Bolzano and Frege or Russel) vs *Logic as Calculus,* as endorsed by Boole, Grassmann, Peirce and Schröder, Peano or Hilbert, etc. (Heijenoort, 1967).

Analytical philosophy turned from epistemology to semantics as the main philosophical issue. Michael Dummett concludes: "The theory of meaning … is the foundation of all philosophy, and not epistemology as Descartes (or Kant, our insertion, M.O.) misled us into believing. Frege's greatness consists, in the first place, in his having perceived this" (Dummett, 1981, p. 669). This development began in fact much earlier and Dummett could have equally well had indicated Bolzano.

By contrast, the axiomatic method and its underlying structuralism attempted to pave the way for a universalization of the applications of formal theoretical knowledge. I believe, said Hilbert, "that everything that can be a subject matter of scientific cognition at all, becomes subject to the axiomatic method, as soon it is ripe, such that a theory can be formulated about it" (Hilbert, 1918, p. 415). And the Norwegian philosopher Foellesdal advocated the thesis "that the so-called hermeneutic method is actually the same as the hypothetic-deductive method applied to materials that are meaningful (e.g. the systems of beliefs and values of human beings in action)" (Føllesdal, 1979).

However, the main road of pure mathematics since the beginning of the 19th century was at the beginning not dominated by the axiomatic and structural approach of Grassmann, Peano and Hilbert, but by the so-called rigor movement of arithmetization, starting with Bolzano, Cauchy, Dirichlet Kronecker, Frege, etc. The rigor movement of arithmetization criticized that the axiomatic characterization of numbers - starting from the series of ordinal numbers - leads to a situation where “every number-symbol becomes infinitely ambiguous” (Russell).

These philosophers, like Frege or Russell, searched for set-theoretical foundations of the number concept. They wanted numbers not merely to verify mathematical formulae, like 7+5=12. They insisted numbers must “apply in *the right way* to common objects” (Russell 1956, p. 9, emphasis added). And how do numbers apply? Frege claims,

> that the content of a statement of number is an assertion about a concept. This is perhaps clearest in the case of the number 0. If I say ‘Venus has 0 moons’, there simply does not exist any moon or agglomeration for asserting anything of; but what happens is that a property is assigned to the *concept* ‘moon of Venus’, namely that of including nothing under it. If I say ‘the King’s carriage is drawn by four horses’, then I assign the number four to the concept ‘horse that draws the Emperor’s carriage’.
> (Frege, 1884, § 46)

But somehow the set-theoretical approach did not work, not least because of the relatively complicated linguistic-logical apparatus. Consider the difficulties with the definition of real numbers or with the nested quantifiers in the definition of continuity after Bolzano, Cauchy and Dirichlet.

And in the end triumphant structuralism and the axiomatic method gained the field everywhere and in all areas of modern science, linguistics, anthropology, science and mathematics. The anthropologist and philosopher Claude Levi-Strauss once said that in science there are two methods only, the reductionist or the structuralist (Levi-Strauss, C., *Myth and Meaning*, Routledge London, chapter 1).

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