



## FROM DESCRIPTIVE HISTORY TO INTERPRETATION AND EXPLANATION – a wave model for the development of mathematics education in Denmark<sup>1</sup>

DA HISTÓRIA DESCRITIVA À INTERPRETAÇÃO E EXPLICAÇÃO – um modelo de onda para o desenvolvimento da educação matemática na Dinamarca

Hans Christian Hansen<sup>2</sup>

 ORCID iD: <https://orcid.org/0009-0002-5085-4920>

### ABSTRACT

How should we write a history of mathematics education? To answer this question, we have to consider why we should want to study this history. For many mathematics teachers – as in the case of the author- it is simply necessary to explore your own profession to improve your teaching. But in the end it would be useful if some people took a more scientific approach and looked for mainlines in the development, tendencies, patterns or even explanations. To do so we have to consider the initial local history of mathematics education in a wider context both in a nationwide and in an international comparison as well in comparison with other school subjects and the history of pedagogy in general. In this paper we explore the possibility to make a wave model of the development of mathematics education in Denmark in the past two centuries, illustrating periodic changes between times with emphasis on understanding and times with emphasis on drill and skills. To which extent can a model like this be explained ?

**Keywords:** *Dig where you stand. Wave model. Danish history.*

### RESUMO

Como devemos escrever uma história da educação matemática? Para responder a esta pergunta, temos que considerar porque deveríamos querer estudar esta história. Para muitos professores de matemática – como é o caso do autor – é simplesmente necessário explorar a sua própria profissão para melhorar o seu ensino. Mas no final seria útil se algumas pessoas adotassem uma abordagem mais científica e procurassem linhas principais no desenvolvimento, tendências, padrões ou mesmo explicações. Para fazer, temos de considerar a história local inicial da educação matemática num contexto mais amplo, tanto numa comparação nacional como internacional, bem como em comparação com outras disciplinas escolares e a história da pedagogia em geral. Neste artigo exploramos a possibilidade de criar um modelo de onda do desenvolvimento da educação matemática na Dinamarca nos últimos dois séculos, ilustrando mudanças periódicas entre tempos com ênfase na compreensão e tempos com ênfase em exercícios e competências. Até que ponto um modelo como este pode ser explicado ?

**Palavras-chave:** *Cave onde você está. Modelo de onda. História dinamarquesa.*

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<sup>2</sup> Dr. scient., Aarhus University, Denmark. Retired. Sønderengen 113, 2870 Dyssegaard, Denmark. E-mail: [hanschrihan0@gmail.com](mailto:hanschrihan0@gmail.com)

## INTRODUCTION

Why should a mathematics teacher take interest in the history of mathematics education? More established answers exist when it comes to the history of mathematics itself. This study has long been considered as part of the general education of the mathematics teacher as well as that of the students of mathematics. And it has been recognized for more than hundred years that the study of the history of mathematics can give the mathematics teacher a wider view on the different approaches to a specific mathematical concept or method.

In fact the historical view was coined in the phrase “the historical-genetic method” around 1880. In Denmark the method was first suggested by the physicist and science teacher at the Askov People’s College, Poul la Cour (1846-1908), in his notes on geometry from 1879, resulting in the published book<sup>3</sup> “Historical Mathematics” in 1881. He explained the method like this: “to let the students follow roughly same path of development as Mankind historically has followed” (Hansen 1985, p. 120). This work was done ten years after E. Haeckel (1834-1919) formulated the biogenetic law (“ontogeny recapitulates phylogeny”) and the inspiration is obvious although la Cour was not keen on the whole Darwinian revolution.

For the historical-genetic method to make sense one must believe in the observation that man basically is the same over historical times, also as a learner. As the biological evolution of man takes thousands of years this observation is not unreasonable as to the physiological side of learning. The same line of reasoning leads to an argument for the study of history of mathematics education. We can probably learn something from our colleagues in history, be they mathematics teachers, educators or text book writers. We might even understand our own personal development as a professional better from a study of the lives of our historical colleagues. And again it seems a natural obligation from the point of view of general education to know something about the history of your profession. And not least, a reflection of the history of mathematics education may hinder absolutism in the didactical discussion and may even reveal useful patterns or periodic developments.

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<sup>3</sup> La Cour built on history books of others like Hankels “Geschichte der Mathematik” (Leipzig 1874), but seems more consciously to have developed the pedagogical aspect; his book was after all called “Historical Mathematics” and not “History of Mathematics”. The case of an independent contribution is supported by the fact that la Cour on request wrote the paragraph on “historisch-genetische Methode” in the German classic E. W. Rein (ed. 1906): *Encyklopädische Handbuch der Pädagogik*.

## 1. THE “DIG WHERE YOU STAND” APPROACH

In 1982 a Danish edition of the originally Swedish (1978) book “Gräv där du står” (Dig where you stand) appeared. It happened while I had my first experience with history writing. “*Sven Lindqvist and his Danish co-workers have had the sympathetic aim of making the workers into bare foot historians*” a reviewer wrote. The idea was that there was important and interesting history in every workplace, and that the professional historians had neglected this local part of history writing, so you had to do it by yourself.

And that was exactly what I was doing. When I took up the position as teacher in mathematics, science and technology at Askov People’s College<sup>4</sup> in 1978, I soon realized that I had to investigate the history of this specific position that I occupied as the sixth person since Poul la Cour had started the whole tradition of having natural science at a people’s college an hundred years earlier in 1878.

I was lucky to find an old cupboard filled with relevant letters, reports and other documents. Combining this with finds in archives and personal interviews I could write the history, which first of all was a biography on Poul la Cour (Hansen 1985, English summary was later put on Wikipedia). It became clear from this study that la Cour – as mentioned above - was a pioneer in the historic-genetic method and in the development of the modern concept of a windmill as a power plant and the relating problems of chemical storing of the energy.

A hundred years later this gave inspiration to my own teaching. As it turned out in the wake of New Math, my students were not particularly interested in the historical approach to mathematics, but wanted to learn some of the New Math, they themselves or their children should use in school or studies. But the whole history of development of windmills turned out to be a good point of departure at a time when Denmark had to look for new energy sources after grassroots and public pressure had terminated all plans of nuclear energy. The “Dig where you stand”- approach turned out to be useful for me as is indicated in figure 1, where we see la Cour’s first windmill from 1891 and the power plant I made (the tower only) with the class of the summer school in 1981 (Hansen 1981).

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<sup>4</sup> A people’s college (“folkehøjskole”) is a boarding school without exams for adults attending it for personal development and general education, which only in few cases include mathematics and science.

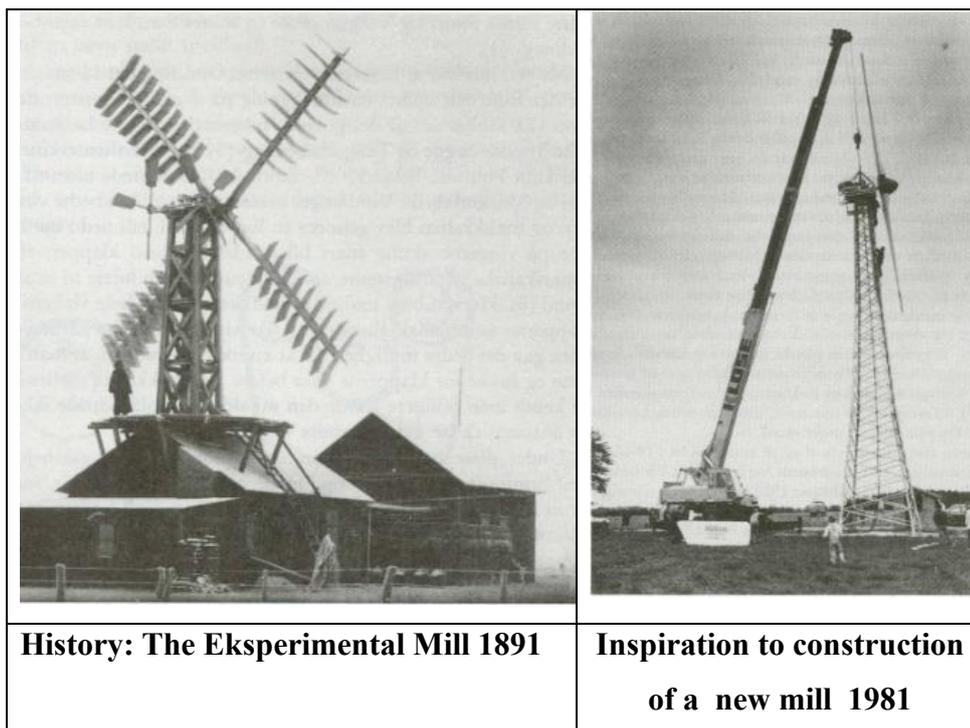


Figure 1

Ten years later I had the same need to dig into the history of a workplace at Brigades Development Centre in the capital Gaborone in Botswana. As an expatriate education officer I needed to get some continuity into the job and it soon turned out from the archives that an American in fact had done a lot of what I now was supposed to do ten years later. So in could build on his work instead of learning from my own mistakes in writing text books in science and mathematics for young people at trade schools (BRIDEC 1991). I include this example to show that sometime an exercise in “dig where you stand” is simply part of doing your job in a professional way and you probably would not even think of it as pursuit in history.

## **2. A MORE SOPHISTICATED APPROACH, A COMMON ENDEAVOR**

Instead of each of us writing our own local history closely connected to our private life and profession it is much more economical to make the pursuit of research a common endeavor. This is possible if we are in a field where there are truths or insights that can be shared by all or many. But it is a fundamental question to which extent research in history can be said to produce truths or it rather develops personal interpretation and meaning.

In any case the present conference is a proof that some people take up the challenge to develop the history of mathematics education as an academic activity and that we believe that

we can do more by sharing and combining our efforts than just to follow our own isolated field and method.

As should be obvious from the introduction I am not a historian by training and my suggestion for a more sophisticated approach to our study is that of the mathematics teacher who has now worked with the history of his subjects for many years. My method from my first work on the history of mathematics education in Denmark was still somewhat guided by the “dig where you stand” approach but now with a more conscious method expressed in the introduction to *“From the grindstone of the mind to the citizens useful tool. Reckoning and mathematics in the people’s school 1739-1958”* from 2002:

My work with this over the past two years has not been guided by conscious theories. I trusted in my many years as a Math teacher on different levels, text book author and mathematician made me a good sounding board for what would be important to include in the story. If something made the sounding board go into resonance; if for instance I wanted to enter into dialogue with my departed colleagues after reading, what they wrote long ago, then I included it... (Hansen 2002, p.2, translated from Danish)

This book has a lot of detailed descriptive history documented by original sources. But it also has a leading idea which is already outlined in the title. And I guess that is the first step forward from the purely descriptive history: to state a thesis and defend it by original sources from history followed up by a critical reflection on the result.

This makes the quest for wider shared truths possible, as a clearly stated thesis concerning the development in one country can give inspiration test if the same thesis holds in another country or even in some sense in the wider history of pedagogy internationally. So this is one thing we can do as an emerging research community: to extract a thesis from local or national development and test it in comparative studies, be they across national borders or across the borders between the different subjects in school.

### **3. THE EMERGENCE OF A WAVE MODEL FOR THE DANISH DEVELOPMENT**

In some cases we could be lucky and find a repeating pattern instead of just one single trend. I will present a wave model for the development of mathematics education in Denmark during the last 200 years as an example. I first suggested it as a footnote in my book from 2002 and gave a more full presentation at a conference in Trondheim in 2003 (Hansen 2004). In fact it came out of my dialogue with history which I used in my classes at the teachers’ college, where I worked. I let the students guess between some quotes from the historical discourse in mathematics education like the following and let students guess about the original dates:

Quotation 1. Mathematics is a tool subject, and we need that children can do their sums. They shall not experiment to realize how two numbers are added or how to multiply fractions.

Quotation 2. Nowadays we often hear complaints that our teenagers cannot do sums, and people add: “We had quite different skills in our days”.

And nobody can deny that that exactly was what you got in the old Arithmetic classes. No time was wasted in letting the children finding their own methods – the teacher did one or at most two sums at the big black board, established the rule and let them do sum after sum of the same kind until they were competent.

Then the change came about. Now children should learn to think and that was the only aim – all the mechanical skills were the Devils work.

After having some guesses I reveal that they are a hundred years apart and the quote 2 is the oldest from 1905 (Lütken, p. 449), whereas quote 1 was made by a senior consultant in the organization Danish Industry as late as in 2000. Of course this is an old game - to find quotes from history all the way back to Socrates that seem very modern. But when I took a look at many more quotes I noticed that it seemed more like a continuing change between something that could be called modern and something more traditional.

I got good help from one of my colleagues in history, Ernst Gehl (1879-1957), a very successful text book writer, who in 1921 for the first time suggested the idea of a wave model:

My point is that ability in arithmetic includes mechanical skills as well as understanding of the present problem. As it is so often the case with many circumstances in life that it follows the wave movement, the same thing has happened in the teaching of arithmetic. Sometimes one has emphasized the one side of arithmetic: the mechanical, but soon after the other: understanding. (Gehl 1921, p. 204)

So here a person eighty years earlier suggested the same idea as was emerging to me from my reading of other sources. Combining Gehl's observation with quote 1 above and other material I got started on the wave theory as illustrated figure 2, a temporal development changing between periods with emphasis on mechanical skills and drill and other periods with emphasis on understanding.



factories made around 1925 about 50 000 cars at their assembly line in Copenhagen. The whole rationality behind the assembly line was that if every worker was perfect in his small field of specialization then the whole factory would be perfect. This was a metaphor that was tempting to transform into pedagogy. In any case the director of the Business College in Copenhagen Marius Vibæk claimed at this time: “The trades and industries demand that the apprentice can do his sums absolutely correct” (Schacht 1971, p. 59). I people listened to representatives from the trades and industries at a time where in 1927 the unemployment was at 22%.

3. Testing showed that the situation was not satisfactory in the reckoning classes in school. From 1917 the Ministry of Education had started testing of 14-years old students across the two different lines in school: the middle school and the continuing primary school. You had to pass several test to be accepted in the more theoretical middle school after grade 5 in primary school and the majority of students continued for a couple of years in the intended more practical continuation of primary school, called “folkeskole”.

The 14-years olds were among other tested in practical reckoning and the test in 1917 revealed not so surprisingly that the level (on a 0 -15 scale) was higher in the middle school (10,4) than in the “folkeskole” (7,7). The real problem was that 11% in the middle school and 37% in the “folkeskole” had done none of the problems in the test correct, and that some schools in general had a very low score. This indicated that something had to be done, change was needed.

4. Some of the teachers writing about their experiences in the journals, I have consulted, also called for change. As one representative I chose Axel Nørby who in 1929 writes in the column “The daily life and work in school” in the main pedagogical journal “Vor Ungdom” (our youth):

The complaint from trades and industries about young people’s lack of proficiency in arithmetic and the great differences in children’s mathematical gifts revealed by experiments indicates that we have to build on other principles if the working conditions and thereby the results shall be improved.

The difficulty in planning the work stems from the difference in the talents; however the demands from trades and industries about proficiency in elementary arithmetic indicate how to solve the problem. It will assist in coining the idea that the child is better fit for its life if it is a master in a small area than a dabbler in everything. (Nørby 1929 p.214)

These are four probable reasons behind the “new reckoning” movement in Denmark in the 1920s. Gehl and his coauthors wrote on this background reckoning books that to a large extent were teacher-proof, organized in numbered lessons and with suggestions for every part of the lesson. It should be emphasized, that things are seldom clear-cut. Gehl of course did not object to understanding in the reckoning classes but as he stated, one have to obey to the rule of the Golden Middle:

Without reckoning skills the children will lose their desire for Reckoning and get to little time to think about the problem and without understanding Reckoning cannot be the training in thinking that it ought to be.

## 5. EXTRAPOLATION OF THE MODEL TO THE 1800S

The model in figure 2 starts with the “old days” as remembered by the teacher Lütken in 1905. There is always a problem with “old days” and “traditional methods”, that it is so easy to say, but what exactly do we mean? As I tried to extrapolate the model into the 1800s I came to a first conclusion that the “old days”, the previous extremity of emphasis on mechanical skills was about 1870, the visible result being the textbook system by Christian Hansen (1830-1910).

This system that later times have labeled traditional with emphasis on mechanical skills, started in the 1860 became extremely popular. In 1882 the 20<sup>th</sup> edition of the book on fractions had sold in 195 000 copies. Fifty years later some of the books had – according to the publisher – sold in 4 million copies, which is a lot in the small country of Denmark with only 2½ million inhabitants around 1900.

I have to study this possible extremity to the “mechanical skills” side around 1870-80 in more detail before I understand it, especially I have to find more information about Chr. Hansen about who almost nothing seems written. As to where it came from I will look closer at the situation in Prussia, where the revolution in 1848 resulted in a conservative reaction also affecting teachers and mathematics in school c.f. Grue-Sørensen (1972) p. 155. This could have affected neighboring Denmark.

What really made me think that there were waves in the 1800s is the fact that the influential text book writer, head master and educator Hans Schneekloth (1812-82) seems to have spearheaded a reform movement against traditional reckoning around 1840 (Figure 3). His main inspiration was from Diesterweg (1790-1866) in Berlin.

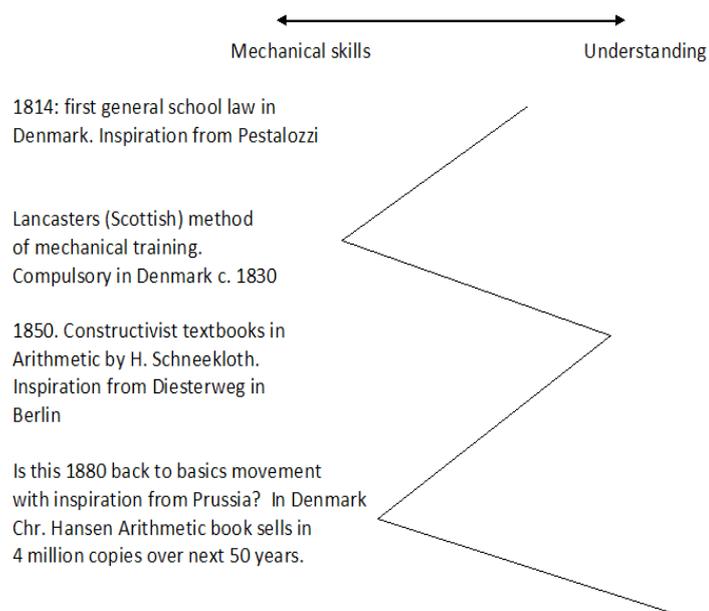


Figure 3

There was enough to rebel against, not least represented by the drilling system in the Bell-Lancaster Method, which was not only dominant in Denmark from about 1820 to 1840 but even instituted by law. The system is well known in most part the western world, so I shall only illustrate it in Figure 4 by a Danish cartoon made by P.C. Klæstrup (1820-1882).

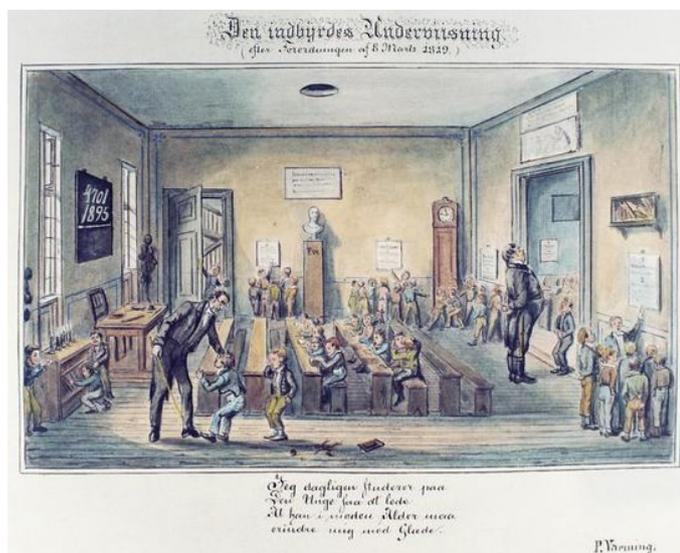


Figure 4

Schneekloth wrote his mental arithmetic text book (*“opgaver til hovedregning”*) in 1841 and in the preface a strong and emotional defense was made for the independent thinking and work of the student:

Mechanical reckoning is when you without insight and understanding follow undigested dictated rules. Anyone using this method is consequently a machine; but Man should not be a machine... To train a child to mindless reckoning - to play with dead digits - is to chain or kill its Spirit. It is intellectual homicide.

The student should under guidance find the operations by himself and reach the general... Others Knowledge is a strange and dead knowledge. Ours is only what our spiritual life has grown in our self... It's best when the student himself forces his way - develop his own solving methods. (Schneekloth 1841, p. 7-10)

Later times would label this view on learning “constructivist”. Certainly he has the modern view of recommending the student to a large extent to find or construct his own methods. In fact we have some of the same formulations in the next extremity to the understanding side in 1905, when Lütken wrote:

The ideal for reckoning classes should be that the children learned to use their innate common sense, had it developed – and not warped by extensive explanations, fixed algorithms and rules – and had the opportunity of trying different methods suggested by themselves and the teacher and subsequently chose the simplest. (Lütken 1905, p. 449)

The question as to where the wave function was at the start of the 1800s we can best look at the time when compulsory school was introduced with a law of 1814. I estimate that the whole spiritual climate at this time and especially in the long planning period that had been before was supportive of understanding in the reckoning classes especially under the influence of Pestalozzi (1746 -1827).

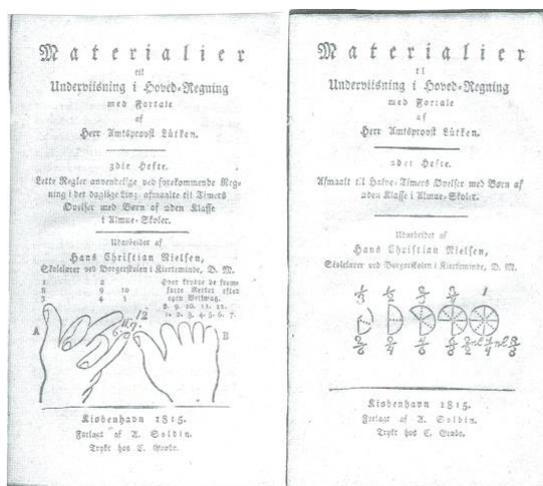


Figure 5. Arithmetic book by Hans Christian Nielsen 1815. Above hand B is written: “let everybody spice the offered dishes according to his own agreeable taste”.

His recommendations to be “anschaulich” (visual instruction) and encourage the independent work of the child had influence on reckoning in Denmark around 1814 as is suggested by the literature (Carlsen 1955). The visual results can be seen from the illustrations on cover (Figure 5) of a new mental arithmetic book at the time (Nielsen 1815). This book has direct references to some of the materials recommended by Pestalozzi (vol. 1 p. 15), and the author writes in the preface:

To be good at reckoning it is not enough to know the rules, as everyone who needs these rules becomes their eternal slave and will be startled by any problem that is not cut to fit the rules... Nay to become a good at written reckoning you need a deep insight in the number system, in the ways quantities can be connected, separated and compared..." (ibid. p. 8)

So although there still is a lot of drill in this arithmetic book, I dare to let it count as part of the documentation of letting the wave model start the 1800s at the understanding side.

## 6. THE WAVE MODEL EXTENDED TO LATE 1900s

It seems feasible to extend the wave model to the second part of the 1900s as illustrated in figure 6. There is no doubt that the intension behind New Math in Denmark was to promote understanding as was the idea in the school law of 1958, and it was not only in Denmark that a back to basics movement had influence about 1980. That the century ended at the understanding side can be seen in the standards of 1995 (Undervisningsministeriet 1995):

Aim 2 for Math: Lessons are planned so students build mathematical knowledge and skills from their own background. Individually and together they shall learn that mathematics is both a tool for problem solving as well as a creative subject

Standards 3. -7. Grade: Students continue to develop their own methods of calculation. Standardized algorithms are introduced if it is a simplification for the student.

Even the chairman of the committee that wrote the standards has acknowledged that their view on learning must be labeled "constructivist" (Jensen 1996, p. 7). A closer look at them would allow us to make the precision "social constructivist" and certainly with a strong emphasis on understanding.

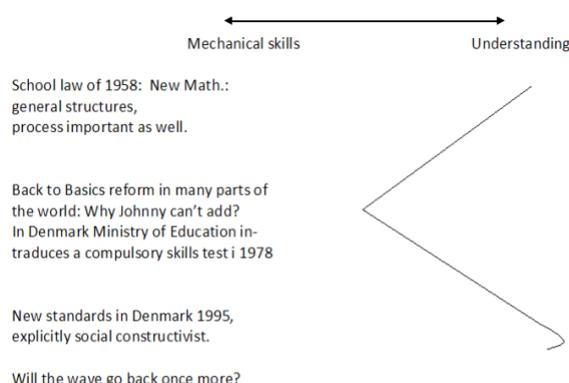


Figure 6

## 7. CRITICAL REFLECTIONS ON THE WAVE MODEL

When a simple model like this comes up in history one has to put on the critical pair of glasses. And not least when it happens in the history of mathematics education with a history writer educated in mathematics – a subject rich in idealized models that in the best case make very rough models of complicated developments involving people and nations and in most such cases is not in any position to produce suitable models. Most developments in the real world are probably too complex and messy to be successfully modeled by present days' mathematics (and mathematicians).

Hence it should be stressed at once that although the model looks like a trigonometric function it does not claim to have a fixed period and amplitude. In fact the “empirical coordinates” behind the function only indicates the extreme situations. At least it should be supplemented by more “empirical coordinates” before we may be justified to use the metaphor of a mathematical function.

Moreover - even if we accept that it seems to be a wave function - we must realize that the empirical input behind the function comes from the *discourse* about mathematics education and that what happened in actual classes may differ from this, both in amplitude and in phase. Knowing the rather heavy work load and the professional loneliness of a 19<sup>th</sup> century village school teacher we may guess that his personal function has smaller amplitude and is delayed in phase. Most of us have experienced that it is possible and even satisfying to participate in one major change, but when it comes to make the same change for the second time we are often more hesitant or even skeptic. Hence it seems difficult to apply the model to individual teachers, to specific classrooms or even schools.

Using something as specific and advanced as a wave model in history is only reasonable and feasible if this completely descriptive model be supported by some kind of explanation, can stimulate the production of such explanations or at least stimulate the discussion of possible explanations. Explanations can take their point of departure in the general history of the time, like with the wide spread use of the drill oriented method of the Bell-Lancaster System for some decennia in Denmark. It would be tempting to find some explanation for introducing this economical method in the disastrous economical situation after the bombardment of Copenhagen in 1807 and the state bankruptcy in 1813. As soon as we start suggesting such explanations we are stimulated to go back and reconsider the model. For this bad economical situation started at a time where our model indicates a start situation with some emphasis on

understanding in reckoning education. Well this could be explained by the very long planning work of several decennia before the school law of 1814 was finally decided on and effectuated. It could well be that the main pedagogical ideas had been developed at an earlier time, so that our descriptive model can be maintained.

The most interesting types of explanation are the ones that can be used every time the wave changes and here we are only suggesting and guessing, which I hope is appropriate at a conference on on-going research. One such tentative explanation is:

*Pure didactical theories never work as well as the inventors and the disciples had expected. Therefore a reaction comes at least with the next generation in the hope of improving the situation.*

Of course an implicit conclusion or rather premise of this explanation is that the truth and the successful, the practicable is in the Golden Middle, a consequence we saw Gehl support and that among others Anna Sfard lately has turned into a theory of learning in mathematics (e.g. Sfard 2003, 11-13).

As to improvements and elaborations of the model one could as suggested by Mogens Niss try it out on other dichotomies like: Citizenship versus labour force; math for the majority versus math for an elite; pure math versus applied math; centralized versus decentralized (standards & exams); active investigating student versus one way teaching; utilitarianism versus general education; formalism versus abstraction and lucidity versus concretization (Ejersbo 2009, p.10).

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