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HISTORY OF MATHEMATICS EDUCATION IN THE ISLAMIC COUNTRIES IN MIDDLE AGES[1]

HISTÓRIA DA EDUCAÇÃO MATEMÁTICA NOS PAÍSES ISLÂMICOS NA IDADE MEDIA

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ABSTRACT

We examine some pedagogical conceptions of the Arabs/Muslims and their practical applications to the teaching of mathematics, the social function assigned to it, its place among the various disciplines taught, the nature of the institutions where mathematics was taught, the status of the students, the teachers and the institutions, the kind of curricula, subjects and textbooks and pedagogies used.
Keywords: History of Mathematics. Mathematic Education. Islamic Countries. Arabic Mathematics. Middle Ages.

RESUMO

Nós examinamos algumas concepções pedagógicas dos árabes/muçulmanos e as suas aplicações práticas ao ensino da matemática, a função social que lhe é atribuída, o seu lugar entre as diversas disciplinas ensinadas, a natureza das instituições onde a matemática era ensinada, o estatuto dos estudantes, dos professores e das instituições, o tipo de currículo, disciplinas e livros didáticos e pedagógicos utilizados.
Palavras-chave: História da Matemática. Educação Matemática. Países islâmicos. Matemática Árabe. Idade Média.

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INTRODUCTION

Very few research works concerning the study of the history of mathematics education in Arab/Islamic Middle Ages have been published; most of them provide "*a useful starting point, [but] also show how much more research remains to be done*" (Djebbar, 2003: 334). The task is difficult, in view of the large period to be covered (more than ten centuries), of the wide variety of countries concerned (from Andalusia to Samarkand), and of the political, economic, linguistic, and cultural transformations recorded throughout the centuries. We examine some pedagogical conceptions of the Arabs/Muslims, the forms of teaching resulting from them, and their practical applications to the instruction of mathematics, the social function assigned to it, its place among the various disciplines taught, the nature of the institutions where mathematics was taught, the status of the students, the teachers and the institutions, the kind of curricula, subjects, textbooks and pedagogies used. The examples we give in this paper are not representative of Arab/Islamic teaching and learning in all areas and for all periods. In fact our data describe some specific forms of mathematics education and some important processes and tools of instruction in some periods, without going into discussing the minor changes that probably occurred.

Texto analítico inédito que apresenta resultados de trabalho de investigação e/ou de reflexão teórico-metodológica. O artigo deve ter **entre 15 e 20 páginas**, abrangendo as referências bibliográficas, legendas, notas de rodapé, quadros e tabelas. Os resumos não devem ser incluídos na contagem de páginas. O artigo deve ser submetido exclusivamente no formato do template do periódico.

1. MATHEMATICS EDUCATION ABSORBED BY THE EDUCATIONAL SYSTEM

All studies on Muslim education in the Middle-Ages show that it was related to worship and religion and "teaching is only stripped of its religious character to a small extent" (Brunschwig, 1947: 352). It follows that, for Muslims, the terms of "science" and "scientist" will cover the widest range of knowledge, primarily religious, and only incidentally secular. The later called “ancient sciences” included philosophy and mathematics and was faithful mainly to the teachings of Ancient Greece. The great physician and philosopher Ibn Sīna (980-1037) included in his *Kitāb al-Shifā* (Book of Healing):

> as inseparable elements of his philosophical teaching a summary of the thirteen books of Euclid's *Elements*, another book on astronomy, a third developing the basis of the theory of numbers, not to be confused with Euclid's Books 7, 8, and 9, nor with Nicomachus's *Introduction to Arithmetic*, and finally a fourth book on music. (Djebbar, 1984: 6-7)

Several centuries later, these same chapters of *Kitāb al-Shifā* were still taught in their original form, reproduced, or quoted in Arabic mathematical textbooks. But gradually, the study of the ancient sciences was restricted to specialists and depended upon the goodwill and mood of the respective ruler. The attitude of orthodox theologians and of the princes under their influence was, however, less clear-cut with regard to mathematics, an art recognized as useful to religion and society. A minimum of knowledge of arithmetic and geometry continued to be part of certain basic teachings; the remaining mathematical topics being henceforth the monopoly of specialists who in all cases cumulated both speculative activities and activities more acceptable to society and more lucrative.

When the works of philosophers were blacklisted, the arithmetic and geometry textbooks were not burnt and continued to be taught. As he put his final touches to his *al-Muqaddima* in 1377, Ibn Khaldūn insisted on the part played by arithmetic and geometry in developing intelligence:

> The crafts, especially writing and calculation, give intelligence to the person who practises them. … Calculation is connected with [writing]. It entails a kind of working with numbers, …, which requires much deductive reasoning. Thus, [the person occupied with it] gets used to deductive reasoning and speculation, and this is what is meant by intelligence. (Ibn Khaldūn, 1988: 331-2)
>
> Geometry enlightens the intellect and sets one's mind right. All its proofs are very clear and orderly. It is hardly possible for errors to enter into geometrical reasoning, because it is well arranged and orderly. Thus, the mind that constantly applies itself to geometry is not likely to fall into error. In this convenient way, the person who knows geometry acquires intelligence. … Our teachers used to say that one's application to geometry does to the mind what soaps do to a garment. It washes off stains and cleanses it of grease and dirt. (Ibn Khaldūn, 1988: 378-9)

Astronomers, whose training required a solid grounding in Euclidian geometry, in the geometry of cones and spheres, as well as in arithmetic and algebra, and whose skills were highly prized by sultans, vizirs, and lesser rulers, contributed towards keeping the teaching of mathematics alive.

Specialists of inheritance tasks also played a crucial part in developing the teaching of arithmetic and algebra.

Once absorbed by the traditional teaching system, ‘secular’ sciences became a minor component of Muslim education. Science teachers accepted the stability of their status and avoided any innovative adventures, because as Brentjes (2002: 65) asserts:

> Their inclusion in the religiously dominated educational scenery and their partnership with the religious sciences granted them stable places for their existence and niches for their

efflorescence. It also subjected them to the same rules of behaviour as the religious sciences in order to be accepted, to impart respectability to students and teachers, and to be transmitted.

2. MATHEMATICS EDUCATION FOR CHILDREN, ADOLESCENTS AND YOUNG ADULTS

In Arab and Islamic cities, the basic education of children, especially boys, have not changed from the early days of Islam till fairly recently. Codified in the tenth century by Ibn Sahnūn (817-870) in his book *Kitāb adāb al-muᶜallimīn* (Deontological Rules for the Teachers), it is given by a master (*mu'addib*) in schools for children (*kuttāb*) or in the learner's home. Most individual testimonials indicate that such studies began at about the age of 5, continued for a period of approximately 5 years, and were essentially devoted to learning the Koran by rote. Ibn Sīna (d.1037) advocated the inclusion of rudiments of writing and reading in the initial training of children but he did not mention mathematics.

2. 1. The instruction of adolescents

While the initial instruction of children was codified, because it was considered crucial for the training of faithful believers, the education of adolescents was not institutionalized. The elite and the educated encouraged their children to pursue their own education. As a rule, the desired training was elementary and comprehensive, focusing on religious subjects, and provided by generalist teachers. Such studies could take between five and ten years and lead to a number of trades such as *mu'addab* (*kuttāb* master) or preacher in a small rural community. All testimonials tend to show that this training devoted some space to a number of chapters on arithmetic deemed necessary to solving inheritance issues (*farā'idh*).

Ibn Sīna suggested that subsequent to basic instruction in the *kuttāb*, the child should move on to specialized training. He recommended:

> deciding to what calling we wish to direct the child. Hence, if we prepare him to be a secretary, we teach him, in addition to language, correspondence and speech writing. Some children are thus directed to the science of reckoning, others to geometry, and others still to medicine (Najjar & Zbidi, 1985: 130-133)

Ibn Sīna states that in his adolescence his father had him attend the classes on Indian arithmetic taught by a greengrocer in Bukhara. Then he assigned to him a professor entrusted with teaching him notions of logic, Euclidian geometry, and astronomy. However, he quickly surpassed his professor's skills and studied Euclid's and Ptolemy's books on his own. When he was 16, he started studying philosophy and medicine.

According to (Bentaleb 1999: 24) the Andalusian Al-Qalasādī (1412-1486) lists in his Rihla (*Account of study travels*) the names of his masters and the disciplines he studied under their aegis. In Seville, his native city, he attended the classes of six masters: the first taught him the Koran and *Sunna* (*Sayings and doings of the Prophet*), as well as arithmetic as developed in *al-Maqālat al-'Arba^c^a* (*The four chapters*) by ibn al-Bannā (d.1321). The fourth master taught him the laws of inheritance and, again, *al-Maqālat al-'Arba^c^a* and the *Talkhīs 'a^c^mal al-hisāb* (*The abridged book on the operations of calculation*) by Ibn al-Bannā. He ended this phase of his education at the age of 24 when he went to Tlemcen (ca. 1436), where he enhanced his knowledge of both the religious sciences and mathematics.

2. 2. The education of young adults

In the Middle-Ages higher education could last many years. Students wishing to receive specialized instruction in mathematics had to organize study travels (*Rihla*), to find specialists. There was no age limit, but at each stage of his journey, the candidate had to demonstrate that he had the scientific skills needed to attend the master's courses.

Often, courses in mathematics were given in *madrasas* or in private homes. This was essentially the master's choice, and depended on his status within the academic community. For example Brunschwig (1947) relates that Ibn Rammah (14th century), in his old age, asked some of his disciples to give lessons in his behalf, in the Kairouan Great mosque every morning, courses in theology and Islamic law, and to spend the rest of the day teaching grammar, *farā'idh,* and *hisāb (reckoning)*. On Friday afternoons, the master gathered his assistants to discuss and address problems.

In his *rihla,* al-Qalasādī (d.1486) relates that he attended, for several years, the classes of Ibn Zaghu (1380-1441) held in the Tlemcen *madrasa al-Ya^c^qubiya*. The professor taught the religious sciences in winter and mathematics and *farā'idh* (*inheritance*) in summer.

Madrasas attached to astronomical observatories were a privileged place for teaching mathematics, as they doubtlessly fostered encounters among the most eminent scientists, surrounded by their disciples and their assistants, and thus created an ideal atmosphere for learning the rational sciences.

Jamshīd al-Kāshī (d.1436) gives in a letter to his father some evidence of scientific life in Samarkand's *madrasa*:

> Every few days, His Majesty the Sultan[3] would be present in the study circle, and when he was there, the study of mathematics would be given priority … One of the examinations of the students is this, that every one who enters the study circle is taken unaware as to what problem will come up, and the people of the *madrasa* was eloquent in repeated investigation of it. (Kennedy, 1986: 194)

3. THE TEACHERS OF MATHEMATICS

Most ulema related either directly or indirectly to mathematics have received some elements of arithmetic, but only scholars specialized *in farā'idh* (*inheritance*), those working in astronomy and time keeping, and also geometers and specialists for commercial transactions were able to teach some mathematical subjects. Many of these scholars, as for example Ibn Khaldūn (d. 1306), would start their professional life as instructors of arithmetic for young boys, sons of wealthy merchants, or of notables. Some others would teach in *kuttāb* to very young boys, charging their parents with fees. Some teachers of mathematics were known to have been professional calculators, consulted by lawyers and judges to help them solve difficult problems of inheritance or conflict between merchants. For example, Balty-Guesdon (1992: 398) reports that Ibn al-ᶜArabī (d.1148) describes the urgent need for this kind of specialist:

> Calculating with small numbers is easy, but with complicated and fractional numbers one has to think for a long period of time and, he might even have to consult a specialist who has to be paid a high price.

Few mathematics teachers enjoyed regular emoluments or pensions paid by the sovereign or by some wealthy donor. While this was the case of those working in prestigious institutions, or in royal or governor's court, or for those teaching as full time professor of law or as assistant in a *madrasa*, all other mathematics instructors had to procure their livelihood by various channels. In fact, this was the case not only for teachers of mathematics, but for all teachers, as reported by Makdisi (1981: 171): *"<Abu Shama> chief complaints were that it was no longer possible as an honest man to make a living teaching in the colleges*."

Many professors of mathematics were at the same time judges, muftis, time-keepers, calendar specialists, or teachers in fields related to these professions. Others would have more

[3] Ulug Beg (1394-1449), ruler of Samarkand, founded the Samarkand observatory in 1425 which was directed by Jamshīd al-Kāshī.

technical vocations: astronomers, astrologers, geometers (*muhandis*), examiners of measures and weights, geographers, tax officers, bookkeepers, copyists of manuscripts, booksellers, etc.

3. 1. Incentives to becoming a teacher of mathematics

Developing rational sciences and teaching them were highly encouraged by Caliph al-Ma'mūn (813-833) who founded *Bayt al-Hikma* (*House of Wisdom*) in Baghdad, Cordoba's Caliphs ᶜAbd ar-Rahmān III (912-961) and al-Hakam II (961-976), Zaragoza's ruler al-Mu'taman ibn Hūd (1081- 1085) who was himself a mathematician, the Mongol Khan Hulagū who built in 1259 the Maragha astronomical observatory headed by Nasīr ad-Dīn at-Tūsī (d. 1274), and the Timūrid ruler and astronomer Ulūgh Beg who founded in 1425 Samarkand astronomical observatory headed by Jamshīd al-Kāshī (d.1436) and its annex *madrasa* directed by Qādhi Zāde ar-Rūmi (ca 1440).

Thes generously financed institutions attracted the very elite of astronomers, mathematicians, and other scholars, Arabs, Muslims, or foreigners alike. They received prestigious visitors as guests on their own in their quest for knowledge, holding conferences for the purpose of presenting original results and new discoveries, followed by discussion and debate. For the adepts of the rational sciences, these were places where to find extensive libraries containing books to be read and copied, as well as masters capable of commentary. For most of these institutions, mathematics was considered to be a propaedeutical knowledge necessary for all the theoretical sciences and practical arts.

3. 2. Master teachers

Most famous teachers of mathematics are those who would attract students from all over the continents, such as did Maslama ibn Ahmad al-Majrītī (d. 1008) the head of the Andalusian mathematicians of Cordoba. No less than five of his disciples became well known teachers of mathematics, astronomy or *farā'idh*. His student Ibn Saffār (d. 1035) also had many students who became teachers of mathematics. Their lines of succession can be traced for many generations in *al-Andalus* and in *al-Maghrib*.

4. MATHEMATICS TEXTBOOKS

A great number of Muslim teachers of mathematics wrote mathematical treatises, commentaries or textbooks. Some of them were general introductions to some field while others were extremely specialized. Some contained an original subject matter apt to change problem solving completely such as al-Khwārizmī’s work on algebra.

Specialized treatises containing high level mathematics (*ummahāt)* will often be evoked and sometime quoted, but their transmission diminished and their copies tended to disappear. Among these *ummahāt,* Arabic translations of Euclid's *Elements,* Apollonius's *Conics*, and Archimedes's *Sphere and Cylinder* were considered compulsory for those who specialized in theoretical geometry, in astronomy and in philosophy. The arithmetical chapters of Euclid's *Elements* and Nicomachus' *Introduction to Arithmetic*, as summarized by Ikhwān as-Safā or by Ibn Sīna, were part of the curriculum for students in philosophy. All these basic standard writings were studied, cut into parts, confronted with newer results, supplemented by original theorems, and eventually replaced by new treatises. These new writings would be abridged into brief texts, which were again commented upon in more voluminous books which were, however, abridged themselves.

For textbooks, Arab authors established a hierarchical order of types: expanded (*mabsūt*), intermediate (*mutawassit*) and abridged (*mukhtasar*).

4. 1. Expanded books

Contain complete theories, with all the necessary propositions and proofs, and illustrated by a large number of examples, as for example Abu Kāmil's *Kitāb al-kāmil fīl al-jabr* (*The complete book in algebra*).

4. 2. Intermediate books

Are "*those in which propositions and expressions are in equilibrium. They can be used by all kind of readers*". Ibn al-Bannā's *Kitāb al-usūl wal muqaddimāt fīl al-jabr* (*The book of the bases and the preliminaries in algebra*) is often placed within this category.

Both expanded and intermediate textbooks were copied, and read by students who studied on their own, calling their professor for assistance when needed.

4.3. Abridged textbooks or epitomes

Have "*terse expression. They are useful for those who, finishing their studies, wish to recall the main propositions, and also for the intelligent beginner capable of catching concepts through concise expression*."

The geometry section of Ibn Sīnā's *ash-Shifa* is considered by its author to be an epitome of Euclid's *Elements* in which, as De Young (2001: 51-2) reports: "*he strives for an economical text in terms of the number of words used while removing some of the standard components of a geometric demonstration*."

Introducing his own arithmetical epitome: *Kashf al-asrār ᶜan ᶜilm hurūf al-ghubār*, Al-Qalasādī wrote: "*I extract this concise and self sufficient book avoiding lengthiness from my book titled "Kashf al-jilbāb", some students will find in it what they need, and more learned persons can use it as a memory aid.*" (Souissi, 1988: 19)

Abridged books had essentially the function of memory aids for scholars who had already studied a subject in detail, sometimes for intellectuals who sought to get a general idea of a given field, and for very gifted students desiring to become initiated to new subject matter. Usually, as soon as an expanded book had been written, its author would write an abridged text summarizing what he considered essential in this work. According to this pattern, abridged books were not supposed to be used as textbooks for beginners, but the pedagogy based on rote learning turned out to change the role of epitomes. According to this new methodology, students would write down some section of an abridged book dictated by their instructor on a wooden slate, learn it by heart, then wipe the slate clean and recite it next day in class. Memorizing prescribed sections of an abridged book, usually consisting of commentaries, was mandatory for attending courses on an advanced level.

This new method was sharply criticized by Ibn Khaldūn (d.1406):

> Scholars often approach the principal learned works of the various disciplines, which are rather voluminous, with an intention to interpret and explain. They abridge these to make it easier [for students] to acquire knowledge of them. This has a corrupting influence upon the process of instruction and is detrimental to the attainment of scholarship. For it confuses the beginner by presenting the final resuls of a discipline to him before he is prepared. This is a bad method of instruction. It also involves a great deal of work for the student. He must study the abridgement's wording carefully, which is complicated to understand because it is crowded with ideas, and try to find out on that basis what the problems of the given discipline might be. Hence, the texts of such abbreviated textbooks are found to be difficult and complicated. A good deal of time must be spent on trying to understand them. (Ibn Khaldūn, 1989: 415-416).

In spite Ibn Khaldūn's critique, this pedagogical pattern flourished in North Africa and in Egypt since the 13th century, and traces were observed in Tunisia even at the beginning of the 20th century. Some abridged handbooks and rhymed prose texts (*urzūza)* became popular and were often the object of commentaries, as for example[4]:

- *Al-Urjūza fi'l jabr wa'l-muqābala* (*Poem on Algebra and al-muqābala*), written by Ibn al-Yāsamīn (d. 1204). It is a rhymed prose with 55 lines, introducing different terms used in algebra and standard resolutions of all six canonical equations.
- *Talkhīs 'a^{c}māl al-hisāb* (*Concise Exposition of Arithmetic Operations*) written by Ibn al-Bannā (d. 1321). It is a short rhetorical presentation of operations on numbers and fractions, proportions and algebra.

4. 4. Commentaries as textbooks

A commentary (*Sharh*) can be of two kinds: either it contains explanations of definitions and propositions, justifications for algorithms, proofs of theorems, numerical examples and solved problems, or it insists on linguistic, syntactical, stylistic, and epistemological aspects of the work's wording. Al-Qalasādi's *Sharh Talkhīs 'A^{c}māl al-hisāb* belongs to the first type, while many of nineteenth century commentaries are of the second type, they are in fact of no help for effective calculations.

The decay of mathematics seems to have been brought about as an ominous and regrettable consequence of this succession of abridged works, commentaries on them, commentaries of commentaries, epitomes of such commentaries, and commentaries on such epitomes ….

4. 5. Subjects of Mathematics

Most subjects found in textbooks concern arithmetic *(hisāb)*:

- o Digital calculation based on finger reckoning and memorization (*al-hisāb al-hawā'i*) as in Abū'l-Wafā al-Buzjānī (940-978): *Kitāb fi mā yahtāju ilayhi al-kuttāb wa'l-^{c}ummāl min ^{c}ilm al-hisāb* (A Book about what is Necessary for Scribes, Dealers, and Others from the Science of Arithmetic).

[4] All information concerning works of Arab/Muslim scholars cited in this paper are listed chronologically in Rosenfeld and İhsanoğlu (2003).

- Sexagesimal as in Sibt-al-Māridīnī (1432-1494): *Raqā'iq al-haqā'iq fi hisāb ad-daraj wa'd-daqā'iq* (Subtilities of Truths on Arithmetic of Degrees and Minutes).
- Indian arithmetic (*al- hisāb al-hindī*): al-Uqlūdisī's *Kitāb al-fusūl fī al-hisāb al-hindī* (The Books on Indian arithmetic).
- Algebra (*al-jabr wal muqābala*) as in Al-Kwārizmī (780-850): *al-Kitāb al-Mukhtasar fi'l-jabr wal muqābala* (Abbreviated Book on the Reckoning of al-jabr and muqābala).

4. 6. Comprehensive arithmetic

Textbooks were used as introductions to the fundamentals of mathematics. They usually begin with an introduction to Indian arithmetic, continued with calculation techniques on positive integers and fractions, and with determining the unknown quantities by using proportions, or 'the method of two errors', or algebra; then follows a section on practical geometry, and a section containing a list of everyday problems with solutions obtained by different methods. As examples of these textbooks:

- Al-Qalasādī (d. 1486): *Sharh Talkhīs ʻa^{c}māl al-hisāb* (Commentary on concise exposition of Arithmetic).
- Bahā ad-Dīn al-^{c}Amilī (1547-1622): *Khulāsat al-hisāb* (Essence of Arithmetic).

4.7. Euclidian geometry (*handasa*) was often taught as an introduction to astronomy courses. The most used textbooks were:

- Nasīr ad-Dīn at-Tūsī (1201-1274): *Tahrīr kitāb usūl al-handasa li-Uqlīdis* (Exposition on "Elements on Geometry" of Euclid).
- as-Samarkandī: *Ashkāl at-ta'sīs* (Propositions of Substantiation).

  In this textbook containing 35 propositions taken from Euclid's *Elements*, as-Samarkandī renewed Euclidian geometrical demonstrative methods, and according to De Young (2001), it was "*widely read and copied for centuries… It was also the subject of a remarkable number of commentaries and supercommentaries of commentaries", in particular by :*
- Qādhī Zāde ar-Rūmī (ca 1440): *Sharh ashkāl at-ta'sīs* (Commentary on "Propositions of Substantiation")

4. 8. An example of mathematics curriculum

Examining Sibt al-Māridīnī's_autobiography, it is possible to have an idea of the textbooks read at al-Azhar mosque-university in the fifteenth century.

Muhammad Sibt al-Māridīnī (1423-1506), a timekeeper of the al-Azhar Mosque in Cairo, wrote no less than 50 treatises on astronomy (sine quadrants, sundials, astronomical tables and prayer times) and at least 23 mathematics textbooks. His prime interest being timekeeping, he wrote an expanded book on sexagesimal arithmetic. His teaching of this subject is attested by his writing three different epitomes on it. Most of the other courses taught at al-Azhar mosque by Sibt al- Māridīnī are based on Ibn al-Hā'īm's works on mathematics and inheritance:

- <u>On heritance sciences</u>: 9 handbooks.
- <u>On arithmetic</u>: 6 textbooks on Indian arithmetic (two of them are commentaries on Ibn al-Hā'īm short works, one is an introductory textbook and three are epitomes of this book), and the last is on *al-hisāb al-hawā'i* (mental and digital calculus).
- <u>On algebra</u>: 6 textbooks (Three of them are commentaries on Ibn al-Hā'īm's short works, one is a commentary of Ibn al-Hā'īm' *Sharh al-Urjuza al-Yasminīya,* followed by two epitomes of it. The last one is an introduction to algebra).

Sibt al-Māridīnī seems to have started teaching Ibn al-Hā'īm with short works, and later taught and wrote his own commentaries and abridged handbooks.

5. ARABIC/ISLAMIC PEDAGOGY FOR MATHEMATICS

Insights into Arab pedagogy can be found in some recently published works, few of them specific for mathematics instruction. Sonja Brentjes (2002: 61) shows that the teaching of mathematics followed the rules and norms that were applied to the religious and the philosophical sciences:

> students were companions of teachers of algebra, the mathematical sciences, timekeeping. (…) Texts on geometry, astronomy or logic could be studied by *qira'a*, in other words, upon the student's personal responsibility under the guidance of a tutor or professor. (…) Texts on arithmetic are reported to have been studied by *sama*^{c}, that is, by means of a teacher's voice. Texts (…) on algebra were memorized. (…) Ibn al-Majdi lectured (*durus*) upon the calculation of inheritance shares and upon arithmetic.

5.1. The status of memorizing in mathematics instruction

Makdisi (1981) pinpoints two aspects which seem to have had noteworthy influence on instruction: memorizing and note-taking.

> In the Middle-Ages, writing books was a function of teaching connected with an oral process of teaching, including dictation and note-taking. Books were meant for students; they were the direct result of the teaching process. (1981: 74)
>
> The development of the memory is a constant feature of medieval education in Islam. (1981: 99).
>
> Memorizing, not meant to be unreasoning rote learning, was <to be> reinforced with intelligence and understanding. (1981: 103)

The entire process of learning had then been organized so as to take into account memorizing as the most important pedagogical means.

- During the lesson, students are seated around the teacher in a *halqa* (a circle of study) to listen the recitation of the day's course by a professor's assistant, then they listen to some commentaries by the professor.
- Going back to his room, the student has to learn the course by heart, he eventually writes it down in a notebook, so it may serves as a reference.
- Some of the student's senior classmates would help him repeat the lesson many times, to make him firm in recalling it.
- To be able to understand the lesson, the student was supposed to study his professor's commentaries on the subject, analyze them and prepare for being quizzed, or even for asking questions.

As examples of this type of teaching, we should like to point out al-Kāshī's testimony in a letter to his father, and Ibn al-Hā'īm's recommendations to his students.

<u>Al-Kāshī's testimony</u>:

> … sometimes in the *madrasa* between <the king Ulug Beg> and one of the students, who asks about a problem from any science, there may be such mutual refutation and give and take as cannot be described. This is because he decreed and directed that until a scientific problem penetrates his mind it is not established, and obsequious flattery should not be indulged in and, if sometimes someone accepts blindly he embarrasses him by saying you are treating us as ignorant. And, for the sake of examination of the problem, he may intentionally insert a mistake into the middle of the argument. As soon as anyone accepts it, he reproaches and shames him. (Kennedy, 1986: 205)

Advanced students had to copy mathematical treatises, and to devote time to study them on their own. Many textbooks were written for that purpose, particularly those on "comprehensive" arithmetic, they contain a great number of numerical examples and several solved problems.

Sometimes authors add some open problems: For example, at the end of *al-Fawa'id al-Baha'iyya fi'l qawa'id al-hisabiyya* (Nice and usuful rules in arithmetic), ibn al-Khawwam (d.1324) lists 33open problems of the Diophantine type. He says: "I do not pretend that I can establish the proof of their impossibility, but I say only that I cannot solve them. Anyone who can do it has competences that I don't have." (Abdeljaouad and Hedfi, 1986: 162).

In his algebra textbook, *Sharh al-Urjūza al-Yasminīyya,* ibn al-Hāʾīm (d. 1312) gives advice to his students:

> When <the author of the poem> says '*follow the scheme carefully*', he expects that the calculator takes some precautions and tries to avoid errors when computing, so that he gets exact results. For that purpose, he has to test the accuracy of the results using adequate rules set for verification of this type of reckoning. … (1989: 77)
>
> Do not trust the apparent easiness of this <numerical> example and its clearness, which might let you think that you have mastered the five steps of the algorithm. … Do not hope that I will detail all the steps needed to resolve each of the problems proposed. … (1989: 78)
>
> These are examples of different kinds proposed to the reader in order to illustrate this proposition. I did not increase the number of examples, however, to make the student bored and tired, but <their number is sufficient> for mastering the technique and exercising one's mind. (1989: 79)
>
> It is evident that <what I have presented here> can be generalized by analogy. Try to generalize, and do not content yourself with memorization, only reproducing the cases treated. What I have presented is in fact applicable to all situations. (1989: 135)

Ibn al-Hāʾīm's demands are examples of good teaching. However, he himself was compelled to write concise résumés of his longer textbooks, and to teach on the basis of his abridged versions. Since the dominant way of teaching was based solely on learning of abbreviated textbooks or didactical poems by heart, professors commented less and less on mathematical aspects of the contents, and more and more on their terminological and stylistic points of view. In fact, the feature of "memorizing" undermined the goal of "comprehension and understanding", and ignorance of mathematics replaced creativity and expertise.

5. 2. Note-taking in mathematics instruction

Many Arab scholars said in their autobiographies that they used to record their own professor's commentaries on the day's lesson either on an erasable board. in order to learn the contents during the night, and then to wipe the board so that its was ready for use next day.

Advanced students, however, were expected to take notes on paper, as suggested in the following text:

> Once the lesson has been learned by heart, the student should write it down from memory. The written record of the lesson is to serve as a reference when recall fails. …
>
> Committing subject matter to writing was considered most important in the process of learning. (Makdisi, 1981: 104)

A large number of mathematical manuscripts still extant shows that most have been annotated on their margins, and interlined. One finds teachers commentaries, student's remarks, solutions of problems, numerical examples not treated in the text, written down in the folios in all directions. In his paper on the Dar-I-Nizāmī in India, De Young (1986: 10) noted that many manuscripts

> were produced with exceptionally widely spaced lines and extraordinarily large margins. <That seems> to show that this was a conscious action on the part of the copyists. It seems likely that this style of copying was intended to facilitate the taking of notes.

De Young also suggests that examining annotations on a manuscript can help determine which parts of it have been effectively studied:

> Typically, the annotations in these manuscripts will occur in the first quarter or third of the volume, then suddenly cease. It would seem that then, as now, instruction did not always cover the entire textbook assigned. (1986: 10)

Abdeljaouad (2002) describes one fascinating 18th century manuscript, written in Istanbul and throwing further light on the function of margins: Its writer added 300 mathematical expressions or solutions of problems represented with North African algebraic symbols in the margins.

5. 3. The dust board

The Arab users of Indian arithmetic associated with it the *takht,* a kind of dust board described in the following terms by al-Hassār, one textbook author of the 12th century:

> It is called 'Ghubār' or Indian. They have given it this name because they used at the very beginning a wooden *lawha* (board) on which thin dust was spread. Then the apprentice reckoner would take a small stick whose form is that of a stiletto, draw the ciphers on the dust and execute the intended calculations. Once the work done, he would wipe up the dust and store it. Efficiency of this tool stems from the fact that one can execute calculations without having to constantly use ink, board and wiping out <of ink>. They used dust instead of ink. *al-Kāmil fil hisāb* (The Perfect in Arithmetic)

It should be noted that the text quoted refers to two kinds of wooden boards, the first one is the *takht* or dust board imported from India and used for a long time in Arab countries, and the second one is the *lawha* which is still being used in *Kuttāb* around Islamic countries. With a cane stick dipped in ink, one writes on a *lawha* covered with soft argil and at the end of the work he wipe it with water.

Using the *takht* as a computing tool, the author of a textbook would write down a rhetorical version of his course illustrated by results taken from the dust board. He would usually precede any use of Indian ciphers by the expression "the image of the result is as follows …". Berggren (1986: 32) explains clearly the process involved in this transposition:

> In the text of his book, Kushyār writes out, in words, all the names of the numbers, and it is only when he is actually exhibiting what is written down on the dust board that he uses the Hindu-Arabic ciphers. A reason for this may be that explanations were considered as text and therefore written in words, like any other text. The examples of what was written on the dust board, however, may have been viewed as illustrations, much like a diagram in a geometrical argument, and they were there to show what the calculator would actually see on the dust board.

Most famous Arabic mathematicians used *takht* and advocated its use for all type of computing. This is the case of Al-Karāji and as-Samaw'al for operations on polynomial expressions represented by tables drawn on the dust board, Omar al-Khayyām and Sharaf ad-Dīn at-Tūsī for calculating solutions of third degree equations, and of North African specialists of algebra who represented equations, and operations on fractions, irrationals, and polynomial expressions by symbols drawn on the dust board.

Associating stick and dust board with reckoning governed the techniques - based on erasing intermediate results - used in Indian arithmetic and later in Arabic algebra. For example, in early Arab/Indian arithmetic textbooks, multiplication of two numbers was presented as a sequence of images copied from the *takht*, inserted in a rhetorical discourse explaining each step of the calculation. This appears clearly in Kushyār ibn Labbān's *Usūl al-hisāb al-hindī* (Principles of Hindu Reckoning). It was also the case later for multiplication of two polynomial expressions, as noted by Berggren (1986: 115):

> As-Samaw'al's procedure is obviously intended to be used on the dust board, where erasure is easy but space is at a premium, and it proceeds by a series of charts. It adapts easily, however, to paper, where erasure is not easy but space is ample.

However, not all Arabic mathematicians were satisfied with these techniques, and from the outset they tried to replace them by methods using paper, pen and ink only, as shown in a's plea in the introduction of part IV of his textbook on Indian arithmetic:

> In this book we state all that is done by Hindi schemes not with *takht* or erasure, but with ink, pen and paper. This is because many a man hates to expose the *takht* between his hands when he finds the need to use this art of calculation, for fear of the misunterpretation of the attendants or whoever may see it. It belittles him, for it is seen between the hands of the misbehaved who earn their living by astrology in the streets. Moreover, he who calculates on it finds it so difficult to reconsider what he has calculated to the extant that in most cases he repeats it, not to mention the exposure of the content to the blowing wind which changes the figures, apart from making the fingers dirty, over and above other things which distort orderliness.
>
> In addition to all that we have said, what we here suggest is simpler and quicker than the arithmetic of the *takht.* Of it we shall show what will be appreciated and considered as a novelty by all who see it. (Uqlīdisī, 1978: 247).

By the end of the 13th century, textbook authors advocated the sole use of paper, pen and ink for all calculations, as for example al-Naysaburī in his *Risāla al-shamsīyya fi'l-hisāb* (Treatise on Arithmetic Dedicated to Shams ad-Dīn).

CONCLUSION

As presented in this study Arab/Islamic mathematical education in the Middle-Ages played first an important role in the society. Then the secular sciences were absorbed in the religious subjects, responding to specific needs of clerics. Teaching mathematics and reschearch were restricted to specialists and depended upon the goodwill of rulers. After having been able to methodically and ingeniously develop many fields of mathematics – Indian arithmetic, algebra, geometry, trigonometry, (…). – the Arabs lost all interest in furthering their knowledge and learning new sciences. They stopped their study trips (*Rihla*) and retired into their own shell, repeating the teaching of ancient and for the most part obsolete subject matters. Teachers of mathematics were comfortably installed in their stable positions within the traditional system but they were no more responding to the needs and concerns of the society in which they lived, and their marginalization stopped the growth of their science and had frozen methods and practices. The interest in exact sciences greatly declined and teaching of mathematics became exceptional.

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