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THE ROLE OF PROJECTIVE GEOMETRY IN TALIAN EDUCATION AND INSTITUTIONS AT THE END OF THE 19TH CENTURY[1]

O PAPEL DA GEOMETRIA PROJETIVA NA EDUCAÇÃO E NAS INSTITUIÇÕES TALIANAS NO FINAL DO SÉCULO XIX

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ABSTRACT

At the end of the 19th century, projective geometry was at the basis of most geometric research in Italy, and of much other research work in Europe. Furthermore, through its close connection with descriptive geometry, it seemed particularly responsive to social and educational needs of the time. In Italy, a reform of the Technical Institutes brought projective geometry into the syllabuses; and Cremona's book Elementi di Geometria Projettiva helped to spread the synthetic method in Italy and in Europe. In this paper we will examine the link between projective geometry and education, from the personal point of view of Luigi Cremona, and from the institutional point of view of technical instruction in schools and universities. Information about the reception of Cremona's book in Europe and some letters from Cremona's estate will help us to understand the scientific climate of that period.
Keywords: Projective Geometry. Luigi Cremona. Italian Technical Institutes. Citadel of Science in Rome.

RESUMO

No final do século XIX, a geometria projetiva estava na base da maioria das pesquisas geométricas na Itália e de muitos outros trabalhos de pesquisa na Europa. Além disso, por sua estreita ligação com a geometria descritiva, parecia particularmente sensível às necessidades sociais e educacionais da época. Na Itália, uma reforma dos Institutos Técnicos trouxe a geometria projetiva para os programas de estudos; e o livro Elementi di Geometria Projettiva, de Cremona, ajudou a difundir o método sintético na Itália e na Europa. Neste artigo examinaremos a relação entre geometria projetiva e educação, do ponto de vista pessoal de Luigi Cremona, e do ponto de vista institucional do ensino técnico em escolas e universidades. Informações sobre a recepção do livro de Cremona na Europa e algumas cartas do espólio de Cremona nos ajudarão a entender o clima científico daquele período..
Palavras chave: Geometria Projetiva. Luigi Cremona. Institutos Técnicos Italianos. Cidadela da Ciência em Roma.

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INTRODUCTION

System of education in the period of the unification of Italy

The first law to regulate Italian schools was the Casati Law of 1859. It was originally passed only for the Kingdom of Sardinia and for Lombardy, and was later gradually extended to the other Italian regions after their annexation had been declared. With few modifications, it was to remain in force until the Gentile Reform of 1923. The Casati Law established the general characteristics of state secondary education, distinguishing between a *classical education*, whose purpose was to provide a literary and philosophical educational background which would prepare young people for higher studies and specifically for academic courses at the State Universities (Art. 188), and a *technical education*, which sought to provide an appropriate general educational background to young people intending a career in the public services, in industry, in commerce and in agricultural management (Art. 272).

Secondary education was divided into a first and a second level. To cover classical secondary education, the Casati Law introduced the Gymnasium and the Lycée (*Ginnasio-Liceo*), which were to become the point of reference for the entire Italian secondary education (Vita, 1986, p.2). The Technical School (*Scuola Tecnica*) and the Technical Institute (*Istituto Tecnico*) were set up for technical secondary education.

Pupils entered the Gymnasium and the Technical School after a primary school extended over four years. The Technical School thus covered the same age range as the present-day middle school (11–14) while the Gymnasium lasted for five years and hence included the first two years of high school. The Technical School soon lost its characteristic of being a preparatory school for the Technical Institute (to which access could also be gained from the Gymnasium) and was transformed into a school for general education.

After finishing the Gymnasium, pupils completed their classical education by attending the Lycée for three years before going on to university. Initially, a technical education was shorter than its classical counterpart because, after the three years at Technical School, only a further three years were foreseen at the Technical Institute. However, through an 1871 reform the duration of the Technical Institute was extended to four years and, in some cases, to five.

1. ITALIAN MATHEMATICS AND MATHEMATICIANS

The political unification of Italy in 1861 led to the various areas of Italian mathematics becoming integrated into the context of European research. The most eminent Italian mathematicians were involved, both scientifically and politically, in bringing Italy back to the forefront of international developments in the fields of science and economics. Francesco Brioschi (1834–1897), Enrico Betti (1823–1892) and Luigi Cremona (1830–1903), who had also taken part in the First War of Independence, can be considered, along with Eugenio Beltrami and Felice Casorati, as the period's leading mathematicians. Brioschi, an analyst, re-established the *Annali di Matematica* in 1857; he founded the Polytechnic of Milan, was General Secretary for Public Education and held several other political posts. Betti, who had introduced Galois' Theory to Italy and who worked on applying Riemann's ideas to mathematical physics, became head of the *Scuola Normale* in Pisa in 1865, an institute which produced numerous researchers in mathematics. He was also a member of the Italian Parliament, and in 1884 became a senator.

This was without a doubt an unique period in the scientific history of Italy. Patriotism intertwined with a lay and positivist mentality born with the Risorgimento[3], which was especially widespread amongst the bourgeoisie, gave scientific research a central role which it was not to have again[4]. After a period of just a few decades, Italian mathematics found its place in the vanguard of research in the early twentieth century.

Of special standing in this context were the geometric studies of a synthetic[5] nature. These achieved their greatest development in the *school of geometry* headed by Luigi Cremona. Cremona anchored his work within the classical school of projective geometry, with particular attention given to the ramifications which came about in France with Poncelet and Chasles, and in Germany with Von Staudt, Plücker, Möbius, Steiner and Clebsch. Cremona's research into birational transformations would prove to be the basis for a great deal of successive studies carried out in Italy in algebraic geometry[6]. Cremona developed his research employing *pure* methods, separating geometric properties from analytical ones. Although the excessive purism of this approach was later criticised, it continued, albeit with the toning down of the "*extreme*" positions, to characterise Italian algebraic geometry, which culminated in the famous (Roman)

[3] The Resurgence. This is the period of political and cultural activism leading to the unification of Italy in the 19th century.

[4] In Brigaglia & Masotto, 1982 there is a good illustration of the interaction between politics and science at the time.

[5] The term *synthetic geometry*, beyond its historical and etymological meaning, is used as a synonym of *pure geometry.* The pure methods in projective geometry are also referred to as *graphic.*

[6] Italian algebraic geometry made use of pure projective methods. For instance, the intersections of curves and algebraic surfaces were mostly described and "counted", instead of being calculated by means of systems of equations.

school of geometry at the beginning of the 20th century, amongst whose leading exponents were Federigo Enriques, Guido Castelnuovo and Francesco Severi.

Born in Pavia in 1830, Luigi Cremona graduated with degrees in civil engineering and architecture. In 1860 he was appointed to the Chair of Advanced Geometry at the University of Bologna, where he also taught descriptive geometry. He then moved to Milan in 1866, where he taught graphical statics at the Polytechnic and advanced geometry at the *Scuola Normale* (annexed to the Polytechnic itself to train Technical Institute teachers). Finally, in 1873 he accepted an invitation to transfer to Rome to head the School for Engineers, in whose reconstruction he played a fundamental role. He did not however abandon the teaching of graphical statics and advanced geometry. His political work was noteworthy. In Parliament he was recognised as an authority when it came to issues of education (Tricomi, 1962), and hence Cremona was able to exert an enormous influence over the organisation of mathematical studies in Italy. He became a senator in 1879, was made vice-president of the Senate and in 1898, for just one month, was Minister for Education.

Luigi Cremona was recognised as a 'maestro' (master) on an international level. His wide circle of scientific relationships is demonstrated by the many translations of his papers and books, as well as by the large collection of about 1000 letters sent to him by world-famous mathematicians, which was discovered several years ago in the Department of Mathematics at the University of Rome (Israel & Nurzia, 1983; Menghini, 1986 and 1993).

2. GEOMETRY IN SECONDARY SCHOOLS

In 1867, before the complete unification of Italy, a reform by Education Minister Coppino introduced the use of Euclid's *Elements* as the textbook for the teaching of geometry in the Gymnasium-Lycée. More specifically, Book I was intended for the fifth year of the Gymansium, Books II and III for the first year of the Lycée, and the successive books for the second year of the Lycée.

Luigi Cremona was a member of the commission which brought Euclid's text into the schools. He also contributed in an unofficial way to the Italian edition of Euclid's *Elements*, which was edited by Betti and Brioschi. At first the reform generated some criticism, but it has provided the model for the teaching of geometry right up to the present day (Menghini, 1996). Geometry was seen as "mental gymnastics" and was intended to get the young used to the rigour of reasoning. It was thought that only the pure geometry of the *Elements* could carry out such a task. In truth, it is not difficult to interpret the 1867 reform as a desire to create a new Italian model freed from the foreign textbooks which were then common in Italy. This point of view is also expressed in some of Cremona's letters (see, for example, his letter to Betti, later in this paper).

The successive reform in 1871 concerned the teaching of geometry in the Technical Institute. The reform presumed the explicit introduction of the fundamental principles of projective geometry, which was deemed to be a necessary theoretical preamble to the study of descriptive geometry. Cremona's *Elementi di Geometria Projettiva*, published in 1873, was later written to fulfil this syllabus.

The successful translations of Cremona's book and their reception in France (1875), Germany (1882) and England (1885) demonstrated the fact that a synthetic treatment of projective geometry was appreciated outside of Italy. Although the book was written for Italian secondary school, it was to be adopted mostly at university level outside Italy.

3. FOREIGN BOOKS IN ITALY

Before the unification and until 1867, Italian textbooks were few in number and moreover were hardly rigorous (Pepe, 2006). Most schools in the various Italian states adopted translations of foreign books. Only a few of these books were appreciated by the leading mathematicians (Brigaglia, 2006). With reference to projective geometry, it is worth mentioning the *Éléments de géométrie* by Amiot, which had been translated, with notes and addenda, by G. Novi (Amiot, 1858). Cremona, in the preface to the Italian edition, praised Amiot for having sought to transform the old books in such a way as to help young people take part in the progress achieved by geometry in the previous one hundred years. However, above all he praised Novi for having completed topics "mentioned too briefly" by Amiot, like the method of the projections, the anharmonic ratio, poles and polars, etc.

The introduction of Euclid's *Elements* in schools brought about the disappearance of most of these books, including the widely adopted *Éléments* of A. M. Legendre (Maraschini & Menghini, 1992; Schubring, 2004)[7]. In his letter to Betti of 8. 9. 1869 Cremona wrote:

> People can say what they want but Euclid's is still the most logical and most rigorous system we have: all the successive systems are impure hybrids; in seeking to remove one defect, they fall

[7] In truth, as Schubring's work shows, the translations into Italian of Legendre's text, in the edition edited by Blanchet, continued well beyond 1867.

> into other worse ones and more than anything else they stop being true geometric systems. Legendre suffices as an example and this even though he is the most respectable of the elementary geometry reformists. However, if we cast our minds back to the books used in our schools before 1867, which would now be re-introduced if the syllabuses are modified, who would dare to deny that the introduction of the Euclidean method has been of immense benefit for our schools?

In fact, Legendre's was a book of elementary geometry whose structure was very similar to the *Elements* of Euclid. But some proofs were "simplified" and the author sometimes made use of arithmetic and algebraic notations which hid the pureness of the geometric treatment.

Two years before the syllabuses for the Gymnasium-Lycée came out, in 1865, Cremona himself translated the *Elemente der Mathematik* by Richard Baltzer (1818–1887) into Italian. Baltzer's work covered many areas (arithmetic, algebra, trigonometry,...). With regard to geometry, the book introduced the primary notions and definitions of elementary geometry rather rapidly and then quickly progressed to more demanding current theorems in Euclidean geometry. Projective problems were not covered. Cremona tried in vain to convince his friend Betti, who was a member of the Public Education Council, to declare it useful for the schools. The expression "for secondary schools" did appear on the title page, however (Balzer, 1865). In this case, apparently unconcerned by the intrusion of foreign books, Cremona held that Baltzer’s book was appropriate for the Technical Institutes. He continued to put forward this book even after 1871, holding that the geometric part could be covered in the first two years of the Technical Institute.

In a lecture given at the opening session of the *Association for the Improvement of Geometrical Teaching* and reported in the *Giornale di Matematiche* (1871), Thomas A. Hirst, objecting to the adoption of Euclid's *Elements* in England, referred to the fact that even Cremona recognized the necessity of a different geometry for the scientific curriculum, and confirmed that Balzer's book was used in most Italian Technical Institutes.

Balzer's book was a compendium for teachers, not a book for use by students in school. But this is in line with the complete absence of didactical tools in Italian schoolbooks. Although they were written for use in school, the intended audience was the teacher, rather than the students. The didactical transposition was left completely in the hands of the teacher.

Later, in 1874, another translation of a German book appeared. It was a book of descriptive geometry by Wilhelm Fiedler (1832–1912). Although it was written for the *Technische Hochschulen*, which are university level schools in Germany, the Italian edition was explicitly translated and adapted for use at the secondary school level, in the Technical Institutes of the Italian Kingdom.

It was certainly appropriate to have Fiedler's book alongside Cremona's Projective

Geometry in the parallel course of descriptive geometry at the Technical Institutes. According to Fiedler, the main scope of the teaching of descriptive geometry is the scientific construction and development of "Raumanschauung"[8]. Fiedler reinforced this point of view in a paper translated and published in the *Giornale di Matematiche*. Fiedler sees a complete symbiosis between descriptive and projective geometry and holds that starting from central projection, which corresponds to the process of viewing, we can develop the fundamental part of projective geometry in a natural and complete way (Fiedler, 1878, p.248). He feels supported by Pestalozzi, who argues that teaching must start with intuition. Fiedler sees these strategies as the best method for the reform of geometry teaching at all levels.

The position of Fiedler was very close to Cremona's[9]. Fiedler never mentions Cremona in his paper, but in a letter to Cremona, at the beginning of 1873, he praises his book and the simple way in which Cremona introduces the topics. Furthermore, asking for information about Italian technical education, he adds "...my interest is in this scientific and educational organization of Italy, in the foundation of unity through the school for a new generation. (Fiedler to Cremona, 1873, in Knobloch & Reich 2017)".

4. INSTITUTIONS: TECHNICAL INSTITUTES IN ITALY

In 1871, Minister Castagnola issued new syllabuses for Technical Institutes. The Technical Institutes had developed considerably after the unification of Italy, having undergone various reforms since 1860, all of which recognised the necessity of the separate developments of humanist and technical education, with an eye to the model of the German *Realschulen* (Morpugo, 1875, XXVI and on; Ulivi, 1978).

Coming after the syllabuses introduced by Minister Coppino in 1867, which incorporated the study of Euclid's *Elements* in the classical education, the 1871 syllabuses represented a further significant development for the teaching of mathematics.[10] The reform embodied in the syllabuses recognised the need for a general literary and scientific education in technical education too and instituted a Physics-Mathematics Section (*Sezione fisico – matematica*). This section consisted of classes in which the scientific topics, particularly mathematics and physics, would receive greater attention. Since this section did not have the

[8] Space perception.
[9] Cremona was inspired by Fiedler in his first University course of Descriptive Geometry in 1860.
[10] Technical Schools depended at that time on the Ministry of Agriculture, Industry and Commerce.

aim of qualifying students to go into the professions, and permitted university entrance, it could be seen as the scientific alternative to the Lycée (the future *liceo classico*). The existence of this section could be considered a victory for the new working middle-classes (Vita, 1986, p. 38), brought about by the pressing consequences of progress of science and technology and on the wave of a mainly positivist philosophy (Carrara, 1966). The *liceo scientifico* was to be introduced much later in 1923, in a climate which was decidedly less favourable to scientific culture. At that time, the stimulus given to science by the Unification was completely lost in favour of an academic life which became increasingly self-confined.

The preface to the syllabuses of 1871 refers to the foundations of mathematics and demonstrates the deep mathematical knowledge[11] of the authors. It established that the mathematics taught in the Technical Institutes should promote "useful and not remote applications" (Ministero etc., 1871), and at the same time "enhance the faculty of reasoning"; and hence the methods used to present it had to be "rigorously precise". Nevertheless, the model was not Euclid's Elements, but was instead provided by the "new doctrine of projectivity", which "supplies graphic constructions to solve first and second order problems" in a straightforward way.

The main points are summarized here:

> The first two years of the Technical Institute are common to all the sections and have a general and preparatory character. The study of geometry starts from the first elementary notions (angles, circles, inscribed figures, equality, equivalence and similarity of plane figures) and includes the graphical multiplication of segments, the transformation of area given a base, preliminary notions of solid figures and their measures.
>
> In the second "advanced" two-year phase, the syllabus of the third year is common to the physics-mathematics section and to the industrial section. Geometry includes: the theory of projections of geometric forms (projective ranges and pencils, cross ratio, complete quadrilateral) with its applications to the graphical solution of the problems of first and second degree[12] and to the construction of the curves of the second order, seen as projections of the circle (this requires: projective ranges in a circle; self-corresponding elements of superposed forms); the theory of involution (conjugate points with respect to a circle); the duality principle in the plane; elements of stereometry and the graphic construction of the barycentres of plane figures.
>
> In the fourth year the program is specific for the physics - mathematics section. It provides for the focal properties of conic sections and the projective properties of conics and spheres; the principles of analytic geometry will be founded on the metric relations (by means of the cross ratio) of projective forms.
>
> In the parallel teaching of *descriptive* geometry, the teacher will start from central projection and from the projective properties of figures and will handle the theory of collinearities, of affinities, of similarities, with attention to homology, up to the construction of intersections of surfaces of

[11] The mathematician F. Brioschi was a member of the Council (Consiglio Superiore per l'Istruzione Tecnica) which edited the syllabus. But surely the syllabuses were inspired by Cremona.

[12] Of course the resolution of a problem of the second degree always requires a compass. But every problem can lead to the construction of the self-corresponding elements in two projective pencils, by means of a fixed circle. In particular, the method suggested by the syllabus is that of "false position".

> the second degree. The teachers of mathematics and descriptive geometry shall cooperate, as both are concerned with the projection of geometric figures (Ministero etc., 1871, p. 52-63)

Mathematics occupied six hours per week in the school, while descriptive geometry occupied three hours. The only didactical suggestion was that pupils should be directed to do a lot of practical work and that the teacher should question them individually and help them in solving exercises.

According to Morpurgo (1875, p. 46) the syllabuses were highly commended. They undoubtedly covered a great deal of ground. Indeed, the original aim was to prepare students in the physics-mathematics classes for direct entry to the School for Engineers without having to attend the two-year preparatory course (see Section 3). This was in the end not permitted, and in 1876 a new reform took place, based on proposals coming from teacher council. The physics-mathematics section "preserves its character of school of a general culture, to which the extensive study of Italian letters, that of modern languages, and a strong teaching of the sciences furnish the strength that the humanist education takes from the Greek and Latin literature" (Ministero etc., 1876). The aim of mathematics teaching is again that of enhancing the faculties of the mind while acquiring notions which are fundamental for further studies at the University. The syllabuses were reduced. The teaching of projective and descriptive geometry was unified and appeared only in the fourth year. After two years of plane geometry, and one year of solid geometry and trigonometry, the study of projective geometry was whittled down to the study of the projective ranges and pencils, and of the harmonic properties and projective relationships in a circle. Descriptive geometry was restricted to orthogonal and central projections, which were taught together with equalities, similarities, affinities and perspective collinearities.

5. INSTITUTIONS: THE CITADEL OF SCIENCE

From his arrival in Rome in 1873 onwards, Luigi Cremona was an advocate of a transformation which involved many aspects of the development, even in a 'physical' sense— of scientific studies and, in particular, of mathematical studies in the capital. Cremona accomplished this through the creation of the School for Engineers and with the aim of overcoming the separation of pure and applied science.

Based on experiences in the Polytechnical Schools in northern Italy, this project foresaw transferring a part of the structures and professors of the Faculty of Science from the old 'Sapienza' to a new site at San Pietro in Vincoli, thus setting up a sort of 'citadel of science' in which, along with other disciplines, all teaching of a mathematical nature was brought together in a newly-founded autonomous Institute of Mathematics. This decision was important not only on an institutional level, but also from a strictly conceptual point of view. Indeed, it reflected the close ties between aspects of a theoretical nature and aspects of a 'concrete' nature—linked to the practice of drawing—in the conceptions of that time and in those of Luigi Cremona in particular.

In the new setting, one could find the *School for Engineers,* the *School of Mathematics*, the *Library* and the *School of Drawing and Architecture*. All these schools were part of the University[13]. The position that mathematics should occupy in science is clearly reflected in this "physical" arrangement.

The courses of study in the School for Engineers lasted for three years, after a preliminary two-year period of studies concerning physics and mathematics at the Faculty of Science.

In the last two decades of the nineteenth century, there was a significant increase in the number of courses conducted within the Faculty of Science at the University of Rome. Cremona was also responsible for the establishment, probably for the first time in history, of a course called Projective Geometry (the former Advanced Geometry) designed for the first year of studies at the University. Later, it was his idea to merge the teachings of analytical and projective geometry, this coming about for the first time in Rome in 1888-89. Cremona was to oversee the School for Engineers until his death in 1903.

6. PROJECTIVE GEOMETRY AND EDUCATION

As is well-known, modern projective geometry—which is generally held to begin with the publication in 1822 of the famous *Traité des propriétés projectives des figures* by Jean-Victor Poncelet—came about essentially due to the considerable developments in descriptive geometry, which had taken place previously, in revolutionary France[14]. Gaspard Monge—with his *Géométrie descriptive*, published in 1799—played a considerable role in these developments and contributed to the design of the rising *Ecole Polytechnique* (Grattan-Guinness, 2005). The military and civilian needs met by the *Ecole Polytechnique* through the

[13] In Rome the School for Engineers was annexed to the Faculty of Science. In other Italian cities, as Milan or Turin, these schools were independent from the University.
[14] Some of the considerations contained in this paragraph, as well as in paragraph 6, were developed with L. Dell'Aglio in (Dell'Aglio, Emmer & Menghini, 2001).

training of engineers and administrators were of course not only a French phenomenon. The *Ecole* influenced the development of technical schools in many countries in 19th century Europe, like the *Technische Hochschulen* in Germany (Schubring 1989, p. 180) and the *Politecnici* in Italy.

In those technical schools, mathematics was seen as a fundamental preparatory topic. However, in the engineering schools, the triad 'drawing, geometry (descriptive and projective), engineering schools', promoted geometry not only as a basic theoretical subject, but also as a practical and applied activity.

So, at the end of the 19th century, the study of projective geometry was validated not only on the basis of the flourishing of theoretical research, but also because of its responsiveness to the requirements set by technical education due to the corresponding needs of society.

Moreover, the synthetic approach developed in Italy was able to satisfy those didactic and cultural needs expressed at the time of the introduction of Euclid's *Elements* into schools. The treatment of geometry was pure and did not require recourse to algebraic or analytical instruments. The proofs were rigorous and made reasoning obligatory. Furthermore, the subject was not dominated by foreign authors and was in line with Italian research into geometry.

In 1965 Campedelli (1965, p.227) listed those factors that he considered important in innovating school curricula: the linkage with the university and thus with the developments of research, the linkage with applications and the didactic and formative aims. These factors were explicit in the Bourbakist reform of the 1960s (OECE, 1959, p. 11–14), and more or less present in Klein's curricular reform around 1900, particularly in his *Erlanger Antrittsrede* (Rowe, 1985). It would appear that they were already considered by those such as Cremona, who endeavoured to bring projective geometry into schools.

Because of the considerable innovation in mathematical content, the reform which began with the Technical Institutes in Italy can be compared with the Bourbakist reform of the 1960s, which was centered on linear algebra. In the reform of the 60s, the didactic and formative aims were advocated by the pedagogues. (In particular, Piaget's proposed cognitive structures were considered to correspond to the mathematical structures of Bourbakism, Piaget *et al*., 1955). In our case, the didactic and formative aims were identified with the enhancement of mathematical reasoning through the synthetic approach. There were no specific methodological or didactical tools proposed in connection with the introduction of projective geometry in schools.

7. CREMONA'S PROJECT

In a letter dated 8th September 1869 (Gatto, 1996) Cremona wrote to his friend Betti:

> I am convinced that modern methods, especially Steiner's and Staudt's, are destined to revolutionise the whole of geometric knowledge right from the elements themselves; with these methods even the most elementary things can be dealt with more simply, more originally and more fruitfully. However, it will not be possible to introduce these methods in schools until an elementary book has been written specifically for that purpose: such a book does not exist and at the moment there is nobody who wishes or is able to write one. Until that still far-off day when that radical reform can be carried out, I believe that Euclid is still the best guide for teaching geometry in the Lycée.

In 1873, Cremona published his *Elements of Projective Geometry*[15]. This was an elementary book introducing modern projective methods into schools (this does not mean that the book was written in such a way that a student could read it), but it did not represent that radical reform to which Cremona looked forward in his letter to Betti. The book was specifically aimed at the Technical Institutes. It contained the topics for the third-year course, in accordance with the 1871 syllabuses, and also a part of the topics and graphic constructions intended for the descriptive geometry syllabus. The focal properties of conic sections were to have been covered in the second volume, but this volume was never published because the syllabuses were reduced in 1876.

In the preface, Cremona states that he had deeply desired the reform and that he felt that it was his duty to write a proper book. The stated purpose of the book was to propagate the useful theories of projective geometry in Italian schools. Such theories, as the preface pointed out, could be found both in the works of Euclid and Apollonius, and in those of Chasles and von Staudt. Cremona stated that projective geometry, apart from scientific results and from the usefulness of its applications, also had the advantage of being very easy to learn, since it required very few preliminary notions. This is especially the case if pure methods are adopted, such as those of von Staudt. We also find in this discussion all those factors which, later in the 20th century, were to be put forward as reasons for innovating school syllabuses.

In his work, Cremona set out to go beyond the mere training of future engineers. Even at an institutional level, the intention seemed to be the creation of a scientific secondary school of a high cultural level, which would be able to compete with the *Licei* in the education of Italy's future leading classes.

> Perhaps in the not too distant future, this will be the springboard for the solution to the problem of the teaching of elementary geometry: then (if I am not mistaken) and only then, will we have

[15] The preface is dated 5. 11. 1872.

something with which it will be worth substituting the Euclidean method. (Cremona, 1873, p. V)

Therefore, it is possible that Cremona's book was not only aimed at enhancing the scientific culture in the Technical Institutes, but also was intended to constitute a first experiment in promoting the use of projective geometry methods in the Gymnasium–Lycée as well.

Actually, this never came about and, indeed, the contrary occurred. In the early 1900s, texts based on the *Elements* of Euclid became the norm in all types of school. Veronese's 1897 textbook, written for the first two years of the Technical Institutes and for the 4th and 5th year of the Gymnasium, also remained faithful to the Euclidean treatment of geometry and did not contain topics or problems of a projective nature.

There is not much written evidence regarding the reception of the syllabuses for the Technical Institutes and of Cremona's book, apart from the general appreciation stated by Morpurgo (see Section 2). In the *Giornale di Matematiche*, the only Italian journal addressed to school teachers (Furinghetti & Somaglia, 1992), we find no explicit reference at all, although the editor, G. Battaglini, had been involved in the syllabus' reform of 1876. On the other hand, most problems and papers published in this journal addressed pure projective geometry. Only later, in the year 1875, in volume XIII, p.341, do we find the questions assigned for the final examination of the Technical Institutes[16].

Other books appeared for the 3rd and 4th year of the Technical Institutes (Reggio, 1891 and De Franchis, 1908) in which the syllabus of 1876 was fulfilled (the syllabus was in force until 1923, with little variations). Books of descriptive geometry continued to exist separately from those of geometry. In a 1903 book, (Farisano, 1903) the author wrote that the teaching of descriptive geometry is entrusted to the teacher of "Constructions" because it must be "only technical".

8. THEORY AND APPLICATIONS, RESEARCH AND METHODS

The scientific and didactic connection between projective geometry and descriptive geometry, or between the theoretical and graphical aspects, started to influence Italian thinking as early as the pre-unification period, as can be seen in the 1838 translation of Monge's

[16] The one concerning projective geometry is the following: "Given three tangents and the direction of the diameters of a parabola, find: 1. the points of intersection of the three tangents: 2. other tangents to the curve: 3. the points of intersections of these latter tangents."

Géométrie descriptive. The conceptual aspect is linked to the relationship between "pure mathematics" and "applied mathematics". From this point of view, it is possible to make a clear distinction between 19th century mathematical thought, which is characterised by a general symbiosis between theory and application, and that of the 20th century, which shows a clear division between the two aspects, especially on formalist bases. In geometry, this position was to lead to the complete separation of the teaching of projective geometry from that of descriptive geometry and, in fact, to a complete exclusion of descriptive geometry from faculties other than for engineering and architecture.

In any case, this division came somewhat later in the universities than in the Technical Institutes. In 1935, the foundation of the Faculty of Architecture in Rome saw the substantial participation of mathematicians in its establishment. Right from its opening, courses of descriptive geometry, mathematical analysis and applications of descriptive geometry were taught to architecture students by some of the leading exponents of the Roman mathematics community of the period (dell'Aglio, Emmer & Menghini, 2001).

What clearly emerges from *Elementi di Geometria Projettiva* is the belief of the importance in the inclusion of topics of a theoretical nature involving projective geometry into the curriculum of future engineers—from the Technical Institutes to the Polytechnic Schools—in which the central role nevertheless remains occupied by graphical activities. Making didactic recommendations, for example, Cremona states very clearly in the 'Preface' "Finally, it should be noted that the graphical execution of problems always accompanies the theoretical reasoning for the proof of theorems and the deductions of corollaries. (Cremona 1873, p. XIII)".

This point of view is clearly closely related to *research and methods*: a science which looks at applications cannot be too "pure". In his research, Cremona operated in the domain of algebraic geometry using pure projective methods. However, his geometry was not wholly independent of analytical algebraic support. Even though Cremona did not use coordinates, and used pure reasoning (and intuition) to prove all his propositions, he nevertheless employed several algebraic methods. Thus, he was willing to use the results of algebraic systems, and concepts taken from algebra (order and genus of algebraic forms), if they could enhance the continuity of the geometric treatment .

The same mathematical conception is at the core of his book for the Technical Institutes:

> I therefore gave greater emphasis to graphic rather than metric properties; I used the procedures of Staudt's *Geometrie der Lage* more often than Chasles' Géometrie Supérieure, although I never wholly excluded metric relations, because this would have had an adverse practical effect on teaching. […] I could have copied Staudt by doing without any preparatory notions whatsoever, but in this case my work would have been too long, and I would not have been able to adapt it

> to the students in Technical Institutes, who are supposed to have studied the usual fundamentals of mathematics in their first two years. (Cremona 1873, p. XIII)

For instance, Cremona maintained reference to affine formulation of the theorems as much as possible, considering projections of figures from a plane onto a parallel plane, thus using parallelism and points at infinity. For example, he introduced homothetic (similar) triangles as a significant case of homological (perspective) triangles, or parallelograms as a particular case of quadrilaterals; he used length and sign of a segment, together with similitude, to prove the invariance of cross ratio, in accordance with Moebius' barycentric calculus, rather than basing it on the complete quadrilateral, as von Staudt did.

9. *ELEMENTI DI GEOMETRIA PROJETTIVA* IN EUROPE

In the Italian title page of Cremona's book, the subtitle is "for the Technical Institutes of the Italian Kingdom". In the various translations there are no indications of the level to which the book is addressed. In fact, at that time, particularly for the technical education, the age of the students was not always clear, and the character of the schools could change from place to place. We have seen, for example, that in Italy there had been the idea to permit the students of Technical Institutes to enter the third year of University directly; and in Germany the *Polytechnische Schulen* changed in the course of the 19th century, becoming *Polytechnische Hochschulen* (Schubring, 1989, p.179).

In any case it can be stated that the translations of Cremona's book were principally adopted at a university level.

The first translation of Luigi Cremona's text was into French, carried out by Eugène Dewulf and published by Gauthier-Villars (Cremona, 1875). Dewulf had also translated *Preliminari ad una teoria geometrica delle superficie* (Bologna 1866–67). There are many letters from Dewulf amongst Cremona's papers concerning the translation work, together with a few from Gauthier-Villars. These testify to the importance attributed to the diffusion of Cremona's work in teaching:

> [...] en ce moment, je prépare mes annonces, prospectives et catalogues à l'occasion de la rentrée. Et si je ne puis y comprendre votre important ouvrage, ce sera une année perdue pour la diffusion dans le monde de l'enseignement en France. (Gauthier-Villars to Cremona, 24.8.1875, in Nastasi 1992)
>
> At the moment, I am preparing my advertisements, summaries and reading lists for the new academic year. If I am unable to include your important work in these then it will be a year lost for its dissemination throughout the world of teaching in France

When the second French edition of the book was decided upon, Gauthier-Villars wrote to Dewulf:

> En dehors des ouvrages répondant à des programmes d'enseignement ou d'examen, la France offre peu de débouchés; c'est triste à dire. Par suite, les traductions françaises d'ouvrages étrangers d'un ordre elevé, n'ont d'ordinaire qu'un médiocre écoulement. Ainsi, pendant les premières années, l'édition française de la *Géométrie projective* s'est peu vendue, comme vous le savez, malgré tout son mérite; et on peut dire qu'elle serait loin d'être épuisée, si les pays étrangers, l'Italie elle-même, n'étaient venus nous prendre des exemplaires, après la fin de l'édition italienne.
>
> La difficulté sera encore plus grande, lorsque notre nouvelle édition se trouvera en face de l'ouvrage original italien et de la traduction allemande. Je n'ai pas hésité cependant à vous demander de préparer le travail; et cela pour le motif suivant que je n'exposerai pas à d'autres qu'à un ancien camarade voulant bien me comprendre.
>
> La Géométrie est absolument délaissée en France; elle n'est plus représentée à l'Académie et n'a pas une seule chaire où on la professe. J'ai pensé qu'un des meilleurs moyens de raviver, dans la limite du possible, le goût de cette Science, était de réimprimer l'ouvrage d'un maître, comme celui de M. Cremona.
>
> [...] Je cherche toujours, dans la limite de mes moyens, publier des traductions pouvant développer certains courants d'études dans notre pauvre pays, qui ne lit rien de ce qui se fait à l'Etranger et qui a si grand besoin d'être tenu au courant des productions nouvelles". (Gauthier-Villars to Dewulf, 27. 12. 1882, in Nastasi 1992)
>
> Sadly, France does not offer much opportunity other than for works tailored for the teaching and examination programs. Consequently, the French translations of high-level foreign works usually sell poorly. Hence, as you are well aware, in its first few years the French edition of *Projective Geometry* sold little despite all its merits. It can be said that it would be far from being sold out if not for the fact that foreign countries, Italy included, came to us to obtain some copies after the Italian edition had been sold out. The difficulty will be even greater when our new edition will find itself up against both the original Italian work and the German translation. Nevertheless, it was without hesitation that I requested that you prepare the work. This is because of the following reason which I will not make known to others, except to a long-standing companion who is especially willing to understand my view. Geometry has been totally neglected in France; it is no longer represented at the Academy and there is not a single chair where it is taught. I considered that, as far as possible, one of the best ways to revive a taste for this science was to reprint the work of a master such as Mr. Cremona. [...] I am still trying, to the extent that I can, to publish translations so as to be able to develop such areas of study in our poor nation, which reads nothing of what is taking place abroad and which desperately needs to be informed of current advancements.

The second translation of *Geometria Proiettiva*, into German, was made by R. Treutvetter and published in Stuttgart by J.C. Cotta (Cremona, 1882). As Cremona himself stated in his preface, attempts had already been made to introduce the first elements of projective geometry into the German schools. These texts were specifically aimed at the *Gymnasia*. Cremona mentioned the book by E. Müller in particular, who enunciated the duality principle for projective forms, alongside topics of a metric nature, and presented the complete quadrilateral, polarity (in a circle) and the conic sections (Struve, 1994).

We do not find letters from Treutvetter amongst Cremona’s papers. Hence, little is known about the use made of Cremona's book. However, it is unlikely that it was employed in schools. On the first page, the translator wrote “The Italian original is destined above all to the *technischen Hochschulen* of the Italian Kingdom; but the graphical method used by the author in his well-known masterly way will surely also find friends in German institutions.”.

The term "Hochschulen" does not seem correct, because "Hochschulen" are universities, while the Technical Institutes were secondary schools. But in 1882, Cremona's book, being no longer in accordance with the 1876 syllabus for the Technical Institutes, had surely shifted to universities.

The English translation was made by Leudesdorf on the basis of the German and French translations (Cremona, 1885). This translation, complete with a chapter on the focal properties of conics, was definitely aimed at university students (in particular those at Oxford), as we can read from the letter of B. Price, from the Clarendon Press in Oxford, to Cremona, dated May 26 1884:

> "My dear Professor Cremona,
>
> I refer to the proposal made to you by many mathematicians of Oxford, when you were on your visit here to Prof. Sylvester, I have now to say that the unanimous wish is that an English translation of your book on Projective Geometry should be published as soon as possible as the demand for such work is great, and there is no English treatise fit to supply at. You will remember that when the question was under discussion, two plans were proposed, viz. (1) to wait for the expected edition of your *whole* work in its Italian language and to translate that. (2) to translate the German edition which was revised and corrected by yourself on the understanding that you would assist the Translator and Editor with the additions and corrections which you have prepared for your new Italian Edition. Now as less time is required for the second plan than for the first the second is preferred: and Mr Leudesdorf, whom you saw and convened with, and who is a much competent person both as a mathematician and as a good linguist is ready to undertake the work." (Nurzia, 1996, p. 199)

Like Italy, England experienced the reintroduction of the Euclidean text in the schools and the polemics that went with it (see Hirst’s address to the *Association for the Improvement of Geometrical Teaching* mentioned in the Introduction). However, it would appear that there were no attempts to modernise teaching by shifting it towards projective geometry. O. Henrici, an assistant and then successor of Hirst at London's University college, wrote:

> "In England, pure geometry is almost unknown, excepting in the elements as contained in Euclid and in the old-fashioned geometrical conics. The modern methods of synthetic projective geometry as developed on the Continent have never become generally known here. [...] The one English mathematician whose mathematical thought is purely geometrical is Dr Hirst, a pupil of Steiner, who in the position he has just relinquished has been able to introduce, as the first, modern geometrical methods into a regular system of professional education[17], whilst showing at the same time by his original work what can be done with these methods" (Presidential Address to section A, Report of the Fifty-Third Meeting of the British Association for the Advancement of Science, Southport, 1884, reported in Nurzia, 1997/98, p. 2)

[17] The reference is to a course held at the University College School of London, which had the main objective of preparing poorer students for entrance to the University of Cambridge.

J. J. Sylvester had adopted Cremona's book in 1884, probably in French, and had promoted, like Hirst, its translation into English. In a letter to Cayley (20. 5. 1884, in Parshall, 1998, p. 251), he wrote:

> "Cremona stayed with me in College for a couple of days. Leudesdorf is to translate his Géométrie Projective (now out of print) into English. I am in the thick of my lectures 3 times a week on this subject and manage to draw on the board sufficiently well for the purposes of instruction all the geometrical constructions required."

In the preface to the English edition Cremona wrote:

> "[...] My intention was not to produce a book of high theories which should to be of interest to the advanced mathematician, but to construct an elementary text-book of modest dimensions, intelligible to a student whose knowledge needs not extend further than the first books of Euclid. I aimed therefore at simplicity and clearness of exposition; and I was careful to supply an abundance of examples of a kind suitable to encourage the beginner, to make him seize the spirit of the methods, and to render him capable of employing them". (Cremona 1885 p. xiv)

It is exactly this "spirit of the methods", this masterly way, in the words of Treutvetter, of using the graphical methods, which seems to have been most appreciated in the didactics of the elementary courses of mathematics. The English translation was reprinted several times, until the early 1900s.

CONCLUSION

The Italian aspiration of forming, around the nucleus of projective geometry, a learned scientific class able to compete with its counterpart in the humanities cannot be considered to have been fulfilled.

On the one hand, as we have seen in the previous sections, there were factors which promoted a reform, even a radical one, of mathematics teaching:

• The moment was favourable: important mathematicians played leading roles in political life and intervened personally not only regarding the school syllabuses but even as far as the textbooks themselves were concerned. The social image of mathematics and of all the scientific disciplines was good since their contribution to the technological development of the nation was recognised.

• Projective geometry possessed some characteristics that appeared appropriate for its introduction in schools, such as the link with academic research, the link with applications through descriptive geometry and the "simplicity" of illustrating it because it required few prerequisites and because of the purity of its reasoning.

On the other hand, there were some significant negative factors:

• The didactical conceptions of the time could not provide for a didactic transposition really suitable for a pupil of high school level. Although it is possible to hold that the teachers were competent and although it is possible to consider that the high schools, the Technical Institutes included, were the schools of the few, there can be no justification for that failure to consider how this subject should be introduced in the schools.

• There was not sufficient time to fruitfully adapt the new topic to school: the fields of application of projective geometry, in particular in its synthetic treatment, did not last long enough.

• The 1871 reform's syllabuses were too vast and ambitious. The fact that the reform was promoted and planned by mathematicians involved in the most advanced research, and the reform being without an involvement on the part of the schools themselves, led to some negative effects; the emphasis was placed on contents and not on questions of didactics and methodology.

• Further, with regard to the idea of transferring topics in projective geometry to the Lycée, this was probably a personal 'dream' of Cremona's. There is no evidence that it was shared by other mathematicians of the time.

However, the idea that a mathematical topic (projective geometry) could represent an element of cultural unification between research, education and society, in an era in which scientific studies were not considered poor cousins, appears interesting.

It was an analogous idea (maybe analogously pretentious, but better supported) that led, around 1960, to a new topic (linear algebra) permeating reforms in teaching all over Europe.

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